

RIGHT-INVARIANT SOBOLEV METRICS H^s ON THE DIFFEOMORPHISMS GROUP OF THE CIRCLE

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ABSTRACT. Let $\text{Diff}^\infty(\mathbb{S}^1)$ denote the Fréchet Lie group of all orientation preserving smooth diffeomorphisms of the circle. In this paper we study the geodesic flow on $\text{Diff}^\infty(\mathbb{S}^1)$ with respect to the right-invariant metric induced by the fractional Sobolev norm H^s for $s \geq 1/2$. We show that the corresponding initial value problem possesses a maximal solution in the smooth category and that the Riemannian exponential mapping is a smooth diffeomorphism from a neighbourhood of 0 in $C^\infty(\mathbb{S}^1)$ onto a neighbourhood of the identity in $\text{Diff}^\infty(\mathbb{S}^1)$.

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1. INTRODUCTION

It is a fundamental observation due to Arnold [2] that the hydrodynamics of an ideal fluid on a compact Riemannian manifold M can be recast as the geodesic flow on the Lie group of volume preserving smooth diffeomorphisms of M . In this general picture there is a great latitude in the choice of an inertia operator which generates a weak right-invariant Riemannian metric on the Lie diffeomorphism group of M .

In the classical papers [2, 11] L^2 -metrics have been used. This determination particularly allows to treat Euler's equation of hydrodynamics as geodesic flows. In the sequel several other equations of physical relevance have been found to arise in this manner [7, 17, 19, 21, 26, 27, 13]. Among these studies Sobolev metrics on the tangent bundle of type H^k with $k \in \mathbb{N}$ and $k \geq 1$ have been investigated in [7, 19, 21, 27]. In [13] an inertia operator on the diffeomorphism group of the circle of the form HD , where H denotes the Hilbert transform and D the spatial derivative, has been considered.

It is the aim of this paper both to unify and to sharpen earlier results for differential operators as inertia operators on the diffeomorphism group of the circle. The essence of the method is to use a Lagrangian formalism for the geodesic flow that leads to evolution equations on the tangent bundle with a smooth propagator which is in addition bounded in the Sobolev norm H^q on bounded subsets of H^q for large q .

Our main result concerns the smoothness of the conjugation of Fourier multiplication operators with right translations in suitable Banach manifold approximation of $\text{Diff}^\infty(\mathbb{S}^1)$, the group of all smooth and orientation preserving diffeomorphisms on the circle. More precisely, let $q \in \mathbb{R}$ with $q > 3/2$ be given and let $\mathcal{D}^q(\mathbb{S}^1)$ denote the set of all orientation preserving homeomorphisms φ of the circle \mathbb{S}^1 such that both φ and φ^{-1} belong to $H^q(\mathbb{S}^1)$. Then $\mathcal{D}^q(\mathbb{S}^1)$ is a Banach manifold and a topological group but it is not a *Lie group*. Indeed, on $\mathcal{D}^q(\mathbb{S}^1)$, right translation $R_\varphi : \psi \mapsto \psi \circ \varphi$ is linear, continuous and hence smooth (for fixed φ); whereas left translation $L_\varphi : \psi \mapsto \varphi \circ \psi$ is only continuous but not in general not differentiable (see [11, 16]). From an analytic point of view, $\text{Diff}^\infty(\mathbb{S}^1)$ may be viewed as an inverse limit of *Banach manifolds*

$$\text{Diff}^\infty(\mathbb{S}^1) = \bigcap_{q > \frac{3}{2}} \mathcal{D}^q(\mathbb{S}^1).$$

The scales of space $\{\mathcal{D}^q(\mathbb{S}^1)\}_{q > 3/2}$ is called a Banach manifold approximation of $\text{Diff}^\infty(\mathbb{S}^1)$.

Let P denotes a general Fourier multiplier of order $r \geq 1$. It extends to a bounded operator from $H^q(\mathbb{S}^1)$ to $H^{q-r}(\mathbb{S}^1)$ for $q \geq 1$. Given $\varphi \in \mathcal{D}^q(\mathbb{S}^1)$, we consider conjugation

$$P_\varphi := R_\varphi \circ P \circ R_{\varphi^{-1}}$$

of P with right translations $R_\varphi(v) := v \circ \varphi$ for $v \in H^q(\mathbb{S}^1)$. Notice that the map

$$(\varphi, v) \mapsto P_\varphi(v), \quad \text{Diff}^\infty(\mathbb{S}^1) \times C^\infty(\mathbb{S}^1) \rightarrow C^\infty(\mathbb{S}^1)$$

is smooth. However, it is not at all obvious that the extension to Sobolev spaces remains smooth.

Problem. *Under which conditions on P is the mapping*

$$(\varphi, v) \mapsto P_\varphi(v), \quad \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1) \rightarrow H^{q-r}(\mathbb{S}^1)$$

smooth?

It is a well-known fact that when P is a differential operator, $P_\varphi(v)$ is a rational expression of v , φ_x and their derivatives (see [11, 12] for instance). In that case, $(\varphi, v) \mapsto P_\varphi(v)$ is smooth for q large enough. However for a general Fourier multiplier we are not aware of any results in this direction. In our main theorem below, we give a *condition on the symbol* of P which ensures that this map is smooth. This answers a question raised in [11, Appendix A], at least in the case of the diffeomorphisms group of the circle. Up to the authors knowledge, these results are new.

Theorem 1. *Let $P = \mathbf{op}(p(k))$ be a Fourier multiplier of order $r \geq 1$. Suppose that its symbol p extends to \mathbb{R} and that for each $n \geq 1$, the function*

$$f_n(\xi) := \xi^{n-1}p(\xi)$$

is of class C^{n-1} , that $f^{(n-1)}$ is absolutely continuous¹ and that there exists $C_n > 0$ such that

$$(1) \quad \left| f^{(n)}(\xi) \right| \leq C_n(1 + \xi^2)^{(r-1)/2},$$

almost everywhere. Then the map

$$(\varphi, v) \mapsto P_\varphi(v), \quad \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1) \rightarrow H^{q-r}(\mathbb{S}^1)$$

is smooth for each $q \in (\frac{3}{2} + r, \infty) \cup \{1 + r\}$.

This condition is satisfied by the inertia operator Λ^{2s} of the Sobolev metric H^s on $\text{Diff}^\infty(\mathbb{S}^1)$ for $s \in \mathbb{R}$ and $s \geq 1/2$.

Corollary 2. *Let $s \in \mathbb{R}$ and $\Lambda^{2s} := \mathbf{op}((1 + n^2)^s)$. If $s \geq 1/2$ then the mapping*

$$(\varphi, v) \mapsto (R_\varphi \circ \Lambda^{2s} \circ R_{\varphi^{-1}})(v), \quad \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1) \rightarrow H^{q-2s}(\mathbb{S}^1)$$

is smooth for any $q \in (\frac{3}{2} + 2s, \infty) \cup \{1 + 2s\}$.

This allows us to prove that the corresponding *weak* Riemannian metric and its geodesic spray are smooth on each Banach manifold approximation $\mathcal{D}^q(\mathbb{S}^1)$ for sufficiently large $q \in \mathbb{R}$. As a corollary and using an argument introduced by Ebin-Marsden in [11], we are able to prove local existence of geodesics on $\text{Diff}^\infty(\mathbb{S}^1)$. We shall prove the following well-posedness result:

¹A real function f is said to be *absolutely continuous* on \mathbb{R} if f has a derivative almost everywhere, the derivative is locally Lebesgue integrable and

$$f(b) = f(a) + \int_a^b f'(\tau) d\tau,$$

for all $a, b \in \mathbb{R}$.

Theorem 3. *Let $s \geq 1/2$ be given and consider the right-invariant Sobolev H^s -metric on $\text{Diff}^\infty(\mathbb{S}^1)$. Then the corresponding geodesic equation has for any initial data in the tangent bundle $T\text{Diff}^\infty(\mathbb{S}^1)$ a unique non-extendable smooth solution $(\varphi, v) \in C^\infty(J, T\text{Diff}^\infty(\mathbb{S}^1))$. The maximal interval of existence J is open and contains 0. Moreover, if $J = (t^-, t^+)$ and either $t^- > -\infty$ or $t^+ < \infty$ then (φ, v) blows up in $H^{1+2s}(\mathbb{S}^1)$, i.e.*

$$\lim_{t \downarrow t^-} \|(\varphi(t), v(t))\|_{H^{1+2s} \times H^{1+2s}} + \lim_{t \uparrow t^+} \|(\varphi(t), v(t))\|_{H^{1+2s} \times H^{1+2s}} = \infty.$$

It is known that for weak Riemannian metrics the exponential mapping fails in general to be a local diffeomorphism, cf. [7]. We clarify the picture to some extent by proving the following result.

Theorem 4. *The exponential mapping \exp at the unit element id for the H^s -metric on $\text{Diff}^\infty(\mathbb{S}^1)$ is a smooth local diffeomorphism from a neighbourhood of zero in $\text{Vect}(\mathbb{S}^1)$ to a neighbourhood of id on $\text{Diff}^\infty(\mathbb{S}^1)$ for each $s \geq 1/2$.*

Finally in applications, it is of some interest to study geodesic flows not only on the full group of diffeomorphism but to allow a normalization of solutions by fixing their value at one point. In our setting this means to consider the operator $\dot{\Lambda}^{2s}$ of the homogeneous Sobolev metric \dot{H}^s on $\text{Diff}^\infty(\mathbb{S}^1)/\text{Rot}(\mathbb{S}^1)$.

Corollary 5. *Let $s \in \mathbb{R}$ and $\dot{\Lambda}^{2s} := \text{op}(|n|^{2s})$. If $s \geq 1/2$ then the mapping*

$$(\varphi, v) \mapsto \left(R_\varphi \circ \dot{\Lambda}^{2s} \circ R_{\varphi^{-1}} \right) (v), \quad \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1) \rightarrow H^{q-2s}(\mathbb{S}^1)$$

is smooth for any $q \in (\frac{3}{2} + 2s, \infty) \cup \{1 + 2s\}$.

The plan of the paper is as follows. In Section 2 we recall some well-known facts on the geometry of the Euler equations. Some basic material related to weak Riemannian metrics on the diffeomorphisms group of the circle is collected in Section 3. In Section 4 we provide the proofs of our main results: Theorem 1, Corollary 2 and Corollary 5. Section 5 is devoted to the study of the smoothness of the metric and the geodesic spray on the extended Banach manifolds $\mathcal{D}^q(\mathbb{S}^1)$. In Section 6 we prove local existence and uniqueness of the initial value problem for the geodesics of the right-invariant H^s metric on $\text{Diff}^\infty(\mathbb{S}^1)$ and discuss the blow-up problem, while in Section 7 we deal with the Riemannian exponential mapping and discuss the problem of geodesic distance. In Section 8 we extend our study to the homogeneous space $\text{Diff}^\infty(\mathbb{S}^1)/\text{Rot}(\mathbb{S}^1)$ and prove local existence result for the homogeneous metric \dot{H}^s . Technical lemmas on Fourier multiplier and continuity lemmas are collected in Appendix A and Appendix B, respectively.

2. GEOMETRIC CONTEXT

2.1. Euler equation on a Lie group. A right-invariant Riemannian metric on a Lie group G is defined by its value at the unit element, that is by an inner product on the Lie algebra \mathfrak{g} of the group. If this inner product is represented by an invertible operator $A : \mathfrak{g} \rightarrow \mathfrak{g}^*$, for historical reasons, going back to the work of Euler on the motion of the rigid body, this inner

product is called the *inertia operator*. The *Levi-Civita connection* of such a Riemannian metric is itself *right-invariant* and given by

$$(2) \quad \nabla_{\xi_u} \xi_v = \frac{1}{2}[\xi_u, \xi_v] + B(\xi_u, \xi_v),$$

where ξ_v is the right-invariant vector field on G generated by $v \in \mathfrak{g}$ and B is the right-invariant tensor field on G , generated by the bilinear operator

$$(3) \quad B(u, v) = \frac{1}{2} \left[\text{ad}_u^\top v + \text{ad}_v^\top u \right]$$

where $v, w \in \mathfrak{g}$ and ad_v^\top is the *adjoint* (relatively to the inertia operator A) of the natural action of the Lie algebra on itself given by

$$\text{ad}_v : w \mapsto [v, w].$$

Remark 6. Notice that

$$\text{ad}_v^\top = A^{-1} \text{ad}_v^* A$$

where ad^* is the coadjoint action of \mathfrak{g} on itself, defined by

$$(\text{ad}_v^* m, w) := -(m, \text{ad}_v w)$$

for $m \in \mathfrak{g}^*$ and $v, w \in \mathfrak{g}$.

Given a smooth path $g(t)$ in G , we define its *Eulerian velocity*, which lies in the Lie algebra \mathfrak{g} , by

$$u(t) = R_{g^{-1}(t)} \dot{g}(t)$$

where R_g stands for the right translation² in G . It can then be shown, see e.g. [12] that $g(t)$ is a *geodesic* if and only if its Eulerian velocity u satisfies the first order equation

$$(4) \quad u_t = -B(u, u).$$

This equation for the velocities is known as the *Euler equation*.

2.2. Euler equation on a homogeneous space. The theory of Euler equations on a homogeneous space G/K has been developed in [20]. Consider a non-negative *degenerate* inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{g} and let

$$A : \mathfrak{g} \rightarrow \mathfrak{g}^*$$

be the corresponding inertia operator. Suppose moreover that $\ker A = \mathfrak{k}$, where \mathfrak{k} is the Lie algebra of K , and that the duality pairing is Ad_K -invariant, that is

$$(5) \quad \langle \text{Ad}_k u, \text{Ad}_k v \rangle = \langle u, v \rangle,$$

for all $k \in K$ and $u, v \in \mathfrak{g}$. Then A induces a right G -invariant Riemannian metric on the space G/K of right cosets $(Kg, g \in G)$.

Remark 7. If the subgroup K is connected, which we suppose below, the condition (5) is equivalent to the following

$$(6) \quad \langle \text{ad}_w u, v \rangle = -\langle u, \text{ad}_w v \rangle,$$

for all $w \in \mathfrak{k}$ and all $u, v \in \mathfrak{g}$.

²We use the same notation R_φ for this diffeomorphism as well as for its tangent map.

In that case, an “Euler equation” describing the geodesic flow on the homogeneous space G/K , can be defined as a special case of the Hamiltonian reduction with respect to the subgroup K action (see [20]). But, unfortunately, there is no useful *contravariant* formulation of this equation similar to the “genuine Euler equation on a Lie group” (4). Indeed, in this case, the Eulerian velocity is only defined *up to a path* in K (see [29] for a recent survey on the subject).

These difficulties clear away if the coset manifold G/K possesses an additional structure in the following sense. Assume that there is a closed subgroup H of G such that the restriction to H of the canonical projection $G \rightarrow G/K$ is a diffeomorphism. Then we may identify the Lie group H , endowed with the right-invariant metric with the Riemannian coset manifold G/K . If such a subgroup H exists, the study of a degenerate, Ad_K -invariant inner product on \mathfrak{g} with kernel \mathfrak{k} reduces to the study of an Euler equation on the Lie algebra \mathfrak{h} of H .

Remark 8. Notice that the restriction of the projection $H \rightarrow G/K$ is a group morphism if and only if K is a normal subgroup of G .

Proposition 9. *Suppose that H and K are closed subgroups of G such that the canonical projection $H \rightarrow G/K$ is a diffeomorphism. Let \mathfrak{g} , \mathfrak{h} , and \mathfrak{k} denote the Lie algebras of G , H , and K , respectively. Assume further that $A : \mathfrak{g} \rightarrow \mathfrak{g}^*$ satisfies the following hypotheses:*

- (i) $\ker A = \mathfrak{k}$,
- (ii) $\text{ad}_w^* \circ A = A \circ \text{ad}_w$, for all $w \in \mathfrak{k}$.

Then, A induces a right-invariant, Riemannian metric on the homogeneous space G/K , identified with H . Moreover, the bilinear operator

$$B(u, v) = \frac{1}{2} A^{-1} \left[\text{ad}_u^* A(v) + \text{ad}_v^* A(u) \right]$$

is well-defined on $\mathfrak{g} \times \mathfrak{g}$. Its restriction to $\mathfrak{h} \times \mathfrak{h}$ induces the Riemannian connection of the metric on H . The corresponding Euler equation on \mathfrak{h} is given by

$$(7) \quad u_t = -B(u, u) = -A^{-1} \left[\text{ad}_u^* A(u) \right].$$

Proof of Proposition 9. In the situation described, we have

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{h}$$

and \mathfrak{h}^* can be identified with

$$\mathfrak{k}^0 = \{m \in \mathfrak{g}^*; \langle m, w \rangle = 0, \forall w \in \mathfrak{k}\}.$$

The inertia operator induces an isomorphism

$$A : \mathfrak{h} \rightarrow \mathfrak{k}^0 \simeq \mathfrak{h}^*,$$

so that the restriction of the associated inner product on \mathfrak{h} induces a Riemannian structure on H .

Now condition (ii) (and the symmetry of A) leads to

$$\langle \text{ad}_u^* A(v), w \rangle = -\langle \text{ad}_v^* A(u), w \rangle$$

for all $u, v \in \mathfrak{g}$ and all $w \in \mathfrak{k}$. Hence,

$$\text{ad}_u^* A(v) + \text{ad}_v^* A(u) \in \mathfrak{k}^0 = \text{Im } A,$$

and the bilinear operator B is well defined. It can be checked directly that the corresponding right-invariant symmetric linear connection defined by B is compatible with the metric on H . This achieves the proof. \square

3. RIGHT-INVARIANT METRICS ON $\text{Diff}^\infty(\mathbb{S}^1)$

Let $\text{Diff}^\infty(\mathbb{S}^1)$ denote the group of all smooth and orientation preserving diffeomorphisms on the circle. This group is naturally equipped with a *Fréchet manifold* structure. More precisely, we can cover $\text{Diff}^\infty(\mathbb{S}^1)$ with charts taking values in the *Fréchet vector space* $C^\infty(\mathbb{S}^1)$ and in such a way that the change of charts are smooth maps (see [7] or [11] for more details). Since the composition and the inverse are smooth maps for this structure we say that $\text{Diff}^\infty(\mathbb{S}^1)$ is a *Fréchet-Lie group*, cf. [16]. Its Lie algebra $T_{id}\text{Diff}^\infty(\mathbb{S}^1) = \text{Vect}(\mathbb{S}^1)$ is isomorphic to $C^\infty(\mathbb{S}^1)$ with the Lie bracket given by

$$[u, v] = u_x v - u v_x.$$

Notice that $\text{Diff}^\infty(\mathbb{S}^1)$ (like any Lie group) is parallelizable

$$T\text{Diff}^\infty(\mathbb{S}^1) \simeq \text{Diff}^\infty(\mathbb{S}^1) \times C^\infty(\mathbb{S}^1).$$

3.1. Weak Riemannian metrics on $\text{Diff}^\infty(\mathbb{S}^1)$. To define a *right-invariant* metric on $\text{Diff}^\infty(\mathbb{S}^1)$, we introduce an inner product on the Lie algebra $\text{Vect}(\mathbb{S}^1) = C^\infty(\mathbb{S}^1)$. In the present paper, we assume that this inner product is given by

$$\langle u, v \rangle = \int_{\mathbb{S}^1} (Au)v \, dx,$$

where $A : C^\infty(\mathbb{S}^1) \rightarrow C^\infty(\mathbb{S}^1)$ is an invertible *Fourier multiplier* (see Appendix A for precise definitions). By translating the above inner product, we obtain an inner product on each tangent space $T_\varphi\text{Diff}^\infty(\mathbb{S}^1)$

$$(8) \quad \langle \eta, \xi \rangle_\varphi = \langle \eta \circ \varphi^{-1}, \xi \circ \varphi^{-1} \rangle_{id} = \int_{\mathbb{S}^1} \eta(A_\varphi \xi) \varphi_x \, dx,$$

where $\eta, \xi \in T_\varphi\text{Diff}^\infty(\mathbb{S}^1)$ and $A_\varphi = R_\varphi \circ A \circ R_{\varphi^{-1}}$. This family of pre-Hilbertian structures, indexed by $\varphi \in \text{Diff}^\infty(\mathbb{S}^1)$, is smooth because composition and inversion are smooth on the Fréchet Lie group $\text{Diff}^\infty(\mathbb{S}^1)$. This way we obtain a right-invariant, *weak Riemannian* metric on $\text{Diff}^\infty(\mathbb{S}^1)$.

In contrast to finite dimensional Riemannian geometry the topology of the fibre of the tangent bundle is of fundamental importance in the case of infinite dimensional manifolds. It is clear that in the smooth category the pre-Hilbertian structure defined by (8) will not induce the Fréchet topology of the tangent space $C^\infty(\mathbb{S}^1)$. This is why it is called *weak*, because the corresponding topology induced on each tangent space of the Fréchet manifold $\text{Diff}^\infty(\mathbb{S}^1)$ is weaker than the usual Fréchet topology.

The very same is still true if we complete the tangent space with respect to a Sobolev norm. Worse, the *weak Riemannian metric* on $\text{Diff}^\infty(\mathbb{S}^1)$ defined by (8), extends (*a priori*) only to a *continuous* family of pre-Hilbertian structures on the bundle $T\mathcal{D}^q(\mathbb{S}^1)$, provided A is a Fourier multiplier of order $r \geq 1$ and $q \geq r$.

On a Fréchet manifold, only covariant derivatives along curves are meaningful and in general, the existence of a symmetric, linear connection, compatible with a weak Riemannian metric, that is

$$\frac{d}{dt}\langle \xi, \eta \rangle_\varphi = \left\langle \frac{D\xi}{Dt}, \eta \right\rangle_\varphi + \left\langle \xi, \frac{D\eta}{Dt} \right\rangle_\varphi,$$

is far from being granted. Nevertheless, in the situation we consider, the map ad_u^\top is well defined and given by

$$\text{ad}_u^\top v = A^{-1} \left(2(Av)u_x + (Av)_x u \right)$$

for $u, v \in C^\infty(\mathbb{S}^1)$. Hence, one can define

$$B(u, v) = \frac{1}{2} A^{-1} \left[2(Av)u_x + (Av)_x u + 2(Au)v_x + (Au)_x v \right]$$

and check that the expression

$$(9) \quad \frac{D\xi(t)}{Dt} = \left(\varphi, w_t + \frac{1}{2}[u, w] + B(u, w) \right),$$

where

$$\xi(t) = (\varphi(t), w(t)) \in T\text{Diff}^\infty(\mathbb{S}^1) \simeq \text{Diff}^\infty(\mathbb{S}^1) \times C^\infty(\mathbb{S}^1)$$

is a vector field defined along the curve $\varphi(t) \in \text{Diff}^\infty(\mathbb{S}^1)$ and $u = \varphi_t \circ \varphi^{-1}$, defines a right-invariant, symmetric linear connection on $\text{Diff}^\infty(\mathbb{S}^1)$ which is compatible with the metric induced by A .

The corresponding Euler equation on $\text{Diff}^\infty(\mathbb{S}^1)$ is given by

$$(10) \quad u_t = -A^{-1} \{ (Au)_x u + 2(Au)u_x \}.$$

3.2. The geodesic spray. Note that the right hand side of (10) is bilinear in u and of order 1 in the sense that if $u \in H^q(\mathbb{S}^1) \cup C^k(\mathbb{S}^1)$ then

$$A^{-1} \{ (Au)_x u \} \in H^{q-1}(\mathbb{S}^1) \cup C^{k-1}(\mathbb{S}^1).$$

Hence the Euler equation (10) cannot be realized as a dynamical system on any of the Banach spaces $H^q(\mathbb{S}^1)$. Of course it can be realized as an ordinary differential equation on $\text{Diff}^\infty(\mathbb{S}^1)$, but it is not for free to change from Banach spaces to Fréchet spaces, bearing in mind that there is no good practical direct theory for ordinary differential equation on Fréchet spaces. This is why we would wish to work on the *Banach approximations spaces* $\mathcal{D}^q(\mathbb{S}^1)$, rather than on $\text{Diff}^\infty(\mathbb{S}^1)$. It is however quite surprising that in Lagrangian coordinates the propagator of evolution equation of the geodesic flow possesses better mapping properties, provided that

- (1) A and D commute, i.e. A is a *Fourier multiplier*, cf. Lemma 23 in the Appendix A,
- (2) the *order* of A is not less than 1.

In fact, assume that $u(t)$ is a solution to (10) on some time interval J and let $\varphi(t)$ the flow associated with $u(t)$, i.e. φ solves

$$\varphi_t(t) = (u \circ \varphi)(t) = u(t, \varphi(t)), \quad t \in J.$$

Letting now $v := u \circ \varphi$, we clearly have

$$\varphi_t = v, \quad v_t = (u_t + uu_x) \circ \varphi.$$

But by assumption u solves the Euler equation (10). Therefore we find

$$\begin{aligned} v_t &= -A^{-1} \{u(Au)_x + 2(Au)u_x - A(uu_x)\} \circ \varphi \\ &= A^{-1} \{[A, u]u_x - 2(Au)u_x\} \circ \varphi, \end{aligned}$$

where we used that $AD = DA$ to obtain the second identity. Introducing now the operator

$$(11) \quad S(u) := A^{-1} \{[A, u]u_x - 2(Au)u_x\},$$

and recalling that $u = v \circ \varphi^{-1} = R_{\varphi^{-1}}v$, we recognize that v solves

$$v_t = (R_\varphi \circ S \circ R_{\varphi^{-1}})(v).$$

The *second order vector field*, defined in a local chart by

$$(12) \quad (\varphi, v) \mapsto (\varphi, v, v, S_\varphi(v) := (R_\varphi \circ S \circ R_{\varphi^{-1}})(v))$$

is called the *geodesic spray*, cf. [22]. Conversely, assume that (φ, v) solves

$$(13) \quad \begin{cases} \varphi_t = v, \\ v_t = S_\varphi(v), \end{cases}$$

on some time interval J . Then it is not difficult to verify that $u := v \circ \varphi^{-1}$ is a solution on J to the Euler equation (10).

Let us now have a closer inspection of $S(u)$, the spray at $\varphi = id$. Since we presupposed that A be of order $r \geq 1$, the second term $A^{-1}\{(Au)u_x\}$ of the spray is bounded in $H^q(\mathbb{S}^1)$ on bounded subsets of $H^q(\mathbb{S}^1)$, provided $q > (1/2) + r$, cf. Lemma 14.

Although the first term $A^{-1}[A, u]u_x$ of the spray is formally still of order 1, we may hope for some smoothing coming from the commutator $[A, u]$. The case when A is a differential operator justifies this hope. Actually, this turns out to be true not only for differential operators³ but even for general Fourier multipliers, cf. again Lemma 14.

Summarizing, in Lagrangian coordinates the geodesic flow with respect to an inertia operator which is Fourier multiplier of order not less than 1 can be realized as an ordinary differential equation on $T\text{Diff}^\infty(\mathbb{S}^1)$ and its propagator is smooth and bounded in $H^q(\mathbb{S}^1)$ on bounded subsets of $H^q(\mathbb{S}^1)$, provided q is large enough.

4. PROOF OF THE MAIN RESULTS

Before entering the details of the proof of Theorem 1, let us recall some basic definitions and notations. Given two Banach spaces E and F and a m -linear mapping U from the m -fold Cartesian product E^m of E into F . The mapping U is *bounded* iff there is a constant $c > 0$ such that

$$\|U(e_1, e_2, \dots, e_m)\|_F \leq c \|e_1\|_E \|e_2\|_E \cdots \|e_m\|_E.$$

Moreover, the space

$$\mathcal{L}^m(E, F) := \{U : E^m \rightarrow F; U \text{ is } m\text{-linear and bounded}\},$$

³The special case where A is a *differential* operator with constant coefficients has been extensively studied in [6, 7, 12].

endowed with the *operator* norm

$$\|U\|_{\mathcal{L}^m} := \sup \{ \|U(e_1, \dots, e_m)\|_F ; e_j \in E, \|e_j\| \leq 1, j \in \{1, \dots, m\} \}$$

is a Banach space.⁴

According to Lemma 26, the n -th Gâteaux φ -partial derivative of the map $(\varphi, v) \mapsto P_\varphi(v)$ on $\text{Diff}^\infty(\mathbb{S}^1) \times C^\infty(\mathbb{S}^1)$ is given by

$$\partial_\varphi^n P_\varphi(v, \delta\varphi_1, \dots, \delta\varphi_n) = R_\varphi P_n(u_1, \dots, u_n) R_\varphi^{-1}(v),$$

where $u_i = \delta\varphi_i \circ \varphi^{-1}$ ($1 \leq i \leq n$) and P_n is n -multilinear. We will write

$$P_{n,\varphi} := R_\varphi P_n R_\varphi^{-1},$$

which is a $(n+1)$ -multilinear operator from $C^\infty(\mathbb{S}^1)$ to $C^\infty(\mathbb{S}^1)$. Our first task will be to show that if each P_n extends to a bounded multilinear operator from $H^q(\mathbb{S}^1)$ to $H^{q-r}(\mathbb{S}^1)$, then the mapping

$$(\varphi, v) \mapsto P_\varphi(v), \quad \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1) \rightarrow H^{q-r}(\mathbb{S}^1)$$

is smooth.

Lemma 10. *Let P be a Fourier multiplier of order $r \geq 1$ and assume that $q \in (\frac{3}{2} + r, \infty) \cup \{1 + r\}$. Suppose further that*

$$(14) \quad P_n \in \mathcal{L}^{n+1}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1)), \quad P_{n+1} \in \mathcal{L}^{n+2}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$$

for some $n \in \mathbb{N}$, where the operators P_n are defined in Lemma 26. Then

$$P_{n,\varphi} \in \mathcal{L}^{n+1}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1)) \quad \text{for any } \varphi \in \mathcal{D}^q(\mathbb{S}^1),$$

and the mapping

$$\varphi \mapsto P_{n,\varphi} : \mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{L}^{n+1}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$$

is locally Lipschitz continuous.

Proof. (a) Let $\varphi \in \mathcal{D}^q(\mathbb{S}^1)$ be given and choose $v_0, \dots, v_n \in H^q(\mathbb{S}^1)$ with $\|v_j\|_{H^q} \leq 1$. By Lemma 34 there exist increasing continuous functions C_1 and C_2 such that

$$(15) \quad \|R_\varphi\|_{\mathcal{L}(H^{q-r})} \leq C_1(\|\varphi\|_{H^{q-r}}), \quad \|R_{\varphi^{-1}}\|_{\mathcal{L}(H^q)} \leq C_2(\|\varphi^{-1}\|_{H^q}).$$

With this notation and (14) we find that

$$\begin{aligned} \|P_{n,\varphi}(v_1, \dots, v_n) v_0\|_{H^{q-r}} &= \|R_\varphi (P_n(R_{\varphi^{-1}} v_1, \dots, R_{\varphi^{-1}} v_n) R_{\varphi^{-1}} v_0)\|_{H^{q-r}} \\ &\leq C_1(\|\varphi\|_{H^{q-r}}) \|P_n\|_{\mathcal{L}^{n+1}(H^q, H^{q-r})} C_2(\|\varphi^{-1}\|_{H^q})^{n+1}. \end{aligned}$$

This shows that

$$P_{n,\varphi} \in \mathcal{L}^{n+1}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1)).$$

(b) Pick now $\varphi_0, \varphi_1 \in \text{Diff}^\infty(\mathbb{S}^1)$ and set $\varphi(t) := (1-t)\varphi_0 + t\varphi_1$ for $t \in [0, 1]$. From (15) we conclude that

$$\begin{aligned} \sup_{t \in [0,1]} \|R_{\varphi(t)}\|_{\mathcal{L}(H^{q-r})} &\leq C_1(\|\varphi_0\|_{H^{q-r}} + \|\varphi_1\|_{H^{q-r}}), \\ \sup_{t \in [0,1]} \|R_{\varphi(t)^{-1}}\|_{\mathcal{L}(H^q)} &\leq C_2(\|\varphi_0^{-1}\|_{H^q} + \|\varphi_1^{-1}\|_{H^q}). \end{aligned}$$

⁴The above notation is consistent in the case $m = 1$, i.e. $\mathcal{L}^1(E, F) = \mathcal{L}(E, F)$. To further disburden our notation, we set $\mathcal{L}(E) := \mathcal{L}(E, E)$.

Choosing now $v_0, \dots, v_n \in C^\infty(\mathbb{S}^1)$ with $\|v_j\|_{H^q} \leq 1$, we obtain from Lemma 26 that

$$(16) \quad P_{n,\varphi_1}(v_1, \dots, v_n)v_0 - P_{n,\varphi_0}(v_1, \dots, v_n)v_0 = \int_0^1 P_{n+1,\varphi(t)}(v_1, \dots, v_n, \varphi_1 - \varphi_0)v_0 dt.$$

This implies

$$\|P_{n,\varphi_1} - P_{n,\varphi_0}\|_{\mathcal{L}^{n+1}(H^q, H^{q-r})} \leq \sup_{t \in [0,1]} \|P_{n+1,\varphi(t)}\|_{\mathcal{L}^{n+2}(H^q, H^{q-r})} \|\varphi_1 - \varphi_0\|_{H^q}.$$

Similarly, we can find a constant $C > 0$, depending only on the norms of φ_0 and φ_1 in $H^q(\mathbb{S}^1)$ such that

$$\sup_{t \in [0,1]} \|P_{n+1,\varphi(t)}\|_{\mathcal{L}^{n+2}(H^q, H^{q-r})} \leq C.$$

Hence

$$\|P_{n,\varphi_1} - P_{n,\varphi_0}\|_{\mathcal{L}^{n+1}(H^q, H^{q-r})} \leq C \|\varphi_1 - \varphi_0\|_{H^q}$$

and the assertion follows from the density of the embedding $C^\infty(\mathbb{S}^1) \hookrightarrow H^q(\mathbb{S}^1)$. \square

Corollary 11. *Let P be a Fourier multiplier of order $r \geq 1$ and assume that $q \in (\frac{3}{2} + r, \infty) \cup \{1 + r\}$. Suppose further that the operators P_n , defined in Lemma 26, belong to $\mathcal{L}^{(n+1)}(H^q(\mathbb{S}^1); H^{q-r}(\mathbb{S}^1))$ for each $n \in \mathbb{N}$. Then the mapping*

$$(\varphi, v) \mapsto P_\varphi(v), \quad \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1) \rightarrow H^{q-r}(\mathbb{S}^1)$$

is smooth with

$$\begin{aligned} D_{(\varphi,v)}^n P_\varphi(v) [(\delta\varphi_1, h_1), \dots, (\delta\varphi_n, h_n)] \\ = P_{n,\varphi}(\delta\varphi_1, \dots, \delta\varphi_n, v) + \sum_{j=1}^n P_{n-1,\varphi}(\delta\varphi_1, \dots, \widehat{\delta\varphi_j}, \dots, \delta\varphi_n, h_j), \end{aligned}$$

for $n \geq 1$, $\delta\varphi_1, \dots, \delta\varphi_n, h_1, \dots, h_n \in H^q(\mathbb{S}^1)$ where $P_{0,\varphi} := P_\varphi$ and the notation $\widehat{\varphi_j}$ means the term has been omitted.

Proof. (a) We first treat the case $n = 1$. Fix $(\varphi, v) \in \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1)$ and define

$$U(\delta\varphi_1, h_1) := P_{1,\varphi}(\delta\varphi_1, v) + P_\varphi(h_1) \quad \text{for } \delta\varphi_1, h_1 \in H^q(\mathbb{S}^1).$$

Clearly, $U \in \mathcal{L}(H^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$, with

$$\|U(\delta\varphi_1, h_1)\|_{H^{q-r}} \leq c_1 \|\delta\varphi_1\|_{H^q} + c_2 \|h_1\|_{H^q}$$

for some positive constants c_1 and c_2 which depend on P , φ , and v .

Given $\varepsilon > 0$, Lemma 10 ensures the existence of a $\delta > 0$ such that

$$\|P_{\varphi+\delta\varphi_1} - P_\varphi\|_{\mathcal{L}(H^q, H^{q-r})} < \varepsilon,$$

and

$$\|P_{1,\varphi+\delta\varphi_1} - P_{1,\varphi}\|_{\mathcal{L}^2(H^q, H^{q-r})} < \frac{\varepsilon}{1 + \|v\|_{H^q}},$$

provided $\|\delta\varphi_1\|_{H^q} < \delta$. Letting $\varphi(t) := \varphi + t\delta\varphi_1$ and invoking Lemma 26, we find

$$\begin{aligned} & P_{\varphi+\delta\varphi_1}(v+h_1) - P_\varphi(v) - U(\delta\varphi_1, h_1) \\ &= P_{\varphi+\delta\varphi_1}(v) - P_\varphi(v) - P_{1,\varphi}(\delta\varphi_1, v) + P_{\varphi+\delta\varphi_1}(h_1) - P_\varphi(h_1) \\ &= \int_0^1 [P_{1,\varphi(t)} - P_{1,\varphi}](\delta\varphi_1, v) dt + (P_{\varphi+\delta\varphi_1} - P_\varphi)h_1. \end{aligned}$$

Thus we find

$$\begin{aligned} & \|P_{\varphi+\delta\varphi_1}(v+h_1) - P_\varphi(v) - U(\delta\varphi_1, h_1)\|_{H^{q-r}} \\ & \leq \frac{\varepsilon}{1 + \|v\|_{H^q}} \|\delta\varphi_1\|_{H^q} \|v\|_{H^q} + \varepsilon \|h_1\|_{H^q} \\ & \leq \varepsilon (\|\delta\varphi_1\|_{H^q} + \|h_1\|_{H^q}). \end{aligned}$$

This shows that $(\varphi, v) \mapsto P_\varphi(v)$ is Fréchet differentiable and the derivative is given by $U(\delta\varphi_1, h_1)$.

(ii) We now provide the induction step $n \mapsto n+1$. Fix again $(\varphi, v) \in \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1)$ and choose $\delta\varphi_1, \dots, \delta\varphi_{n+1}, h_1, \dots, h_{n+1} \in H^q(\mathbb{S}^1)$. Put further $\varphi(t) := \varphi + t\delta\varphi_{n+1}$. Making use of Lemma 26, we find

$$\begin{aligned} & \left[D_{(\varphi+\delta\varphi_{n+1}, v+h_{n+1})}^n P_\varphi(v) - D_{(\varphi, v)}^n P_\varphi(v) \right] [(\delta\varphi_1, h_1), \dots, (\delta\varphi_n, h_n)] \\ & - \left\{ P_{n+1,\varphi}(\delta\varphi_1, \dots, \delta\varphi_{n+1}, v) + \sum_{j=1}^{n+1} P_{n,\varphi}(\delta\varphi_1, \dots, \delta\widehat{\varphi_j}, \dots, \delta\varphi_{n+1}, h_j) \right\} \\ &= \int_0^1 [P_{n+1,\varphi(t)} - P_{n+1,\varphi}](\delta\varphi_1, \dots, \delta\varphi_{n+1}, v) dt \\ & + [P_{n,\varphi+\delta\varphi_{n+1}} - P_{n,\varphi}](\delta\varphi_1, \dots, \delta\varphi_n, h_{n+1}) \\ & + \sum_{j=1}^n \int_0^1 [P_{n,\varphi(t)} - P_{n,\varphi}](\delta\varphi_1, \dots, \delta\widehat{\varphi_j}, \dots, \delta\varphi_{n+1}, h_j) dt. \end{aligned}$$

Invoking again Lemma 10, we see that, given $\varepsilon > 0$, the expressions on the right hand side of the last equality can be estimated in $H^{q-r}(\mathbb{S}^1)$ from above by

$$\varepsilon (\|\delta\varphi_{n+1}\|_{H^q} + \|h_{n+1}\|_{H^q}),$$

provided $\|\delta\varphi_{n+1}\|_{H^q}$ is small enough. This completes the proof. \square

We are now ready to prove our main results : Theorem 1, Corollary 2 and Corollary 5.

Proof of Theorem 1. The condition on the symbol p of the Fourier multiplier P ensures that

$$P_n \in \mathcal{L}^{n+1}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$$

for $q \in (\frac{3}{2} + r, \infty) \cup \{1+r\}$ and $n \in \mathbb{N}$ (see Appendix A for the details). Now, Theorem 1 results from Corollary 11 of Lemma 10 given below. \square

Proof of Corollary 2. We have to check that the symbol of Λ^{2s} satisfies the hypothesis of Theorem 1. Let $f_n(\xi) := \xi^{n-1}g_s(\xi)$, where $g_s(\xi) := (1 + |\xi^2|)^s$.

Using the Newton formula for the n -th derivative of f_n , it is sufficient to show that

$$\left| \frac{\xi^{k-1} g_s^{(k)}(\xi)}{(1 + \xi^2)^{s-1/2}} \right|$$

is bounded for $k = 1, \dots, n$. This can be checked easily using the fact that

$$g_s^{(k)}(\xi) = \frac{p_k(\xi)}{(1 + \xi^2)^k} g_s(\xi)$$

for $k \geq 1$, where p_k is a polynomial function of degree less than k . \square

Proof of Corollary 5. We have to check that the symbol of $\dot{\Lambda}^{2s}$ satisfies the hypothesis of Theorem 1. Let $f_n(\xi) := \xi^{n-1} g_s(\xi)$, where $g_s(\xi) = |\xi|^{2s}$. The function f_n is clearly of class C^{n-1} and $f^{n-1}(\xi) = |\xi|^{2s}$ satisfies equation (1) of Theorem 1 for $r = 2s$. \square

5. SMOOTHNESS OF THE THE METRIC AND THE SPRAY ON $\mathcal{D}^q(\mathbb{S}^1)$

For a *strong* Riemannian metric on a Banach manifold M , there exists always a unique symmetric covariant derivative compatible with the metric (see [22]). The equations for the *geodesics* of this covariant derivative (curves which minimize the energy functional) correspond to a smooth quadratic second order vector field called a spray. Conversely, given a smooth spray on such a manifold, it induces a symmetric covariant derivative on the Banach manifold, which however may *not be metric*, in the sense that it may not be compatible with any Riemannian metric on M . This is no longer true for a *weak* Riemannian metric on M , in general. The standard proof of the existence of a symmetric compatible covariant derivative involves the Riesz lemma, which is not available for weak metrics.

Nevertheless, we shall prove in this section that the geodesic spray (12) of a right-invariant weak Riemannian metric on $\text{Diff}^\infty(\mathbb{S}^1)$ extends to a smooth spray on the extended Banach manifold $\mathcal{D}^q(\mathbb{S}^1)$ provided that the conjugates of the inertia operator A

$$(\varphi, v) \mapsto A_\varphi(v), \quad \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1) \rightarrow H^{q-r}(\mathbb{S}^1)$$

are smooth. The very same definition of the covariant derivative associated to a spray [22] on a Banach manifold and the origin of this spray ensures then that this covariant derivative is compatible with the weak metric on $\mathcal{D}^q(\mathbb{S}^1)$.

Proposition 12. *Let $m \geq 1$, $r \geq 1$ and $q \in (\frac{3}{2} + r, \infty) \cup \{1 + r\}$. Let further A be a Fourier multiplier of order r . Suppose that*

$$(\varphi, v) \mapsto A_\varphi(v) = R_\varphi \circ A \circ R_{\varphi^{-1}}(v).$$

is of class C^m from $\mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1)$ to $H^{q-r}(\mathbb{S}^1)$ and that A induces an isomorphism from $H^q(\mathbb{S}^1)$ onto $H^{q-r}(\mathbb{S}^1)$. Then the mapping

$$(\varphi, v) \mapsto S_\varphi(v) = R_\varphi \circ S \circ R_{\varphi^{-1}}(v), \quad \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1) \rightarrow H^q(\mathbb{S}^1),$$

is of class C^{m-1} , where

$$S(u) = A^{-1} \{[A, u]u_x - 2(Au)u_x\}.$$

Proof. Let $P(u) := (Au)u_x$ and $Q(u) := [A, u]u_x$. We have

$$S_\varphi(v) = A_\varphi^{-1} \{Q_\varphi(v) - 2P_\varphi(v)\},$$

where the subscript φ indicates the conjugacy by the right translation R_φ in $\mathcal{D}^p(\mathbb{S}^1)$. Although P and Q are smooth operators, these results do not carry over when conjugated with translation in $\mathcal{D}^q(\mathbb{S}^1)$ since for $q > 3/2$ these sets only form topological groups: neither composition nor inversion are differentiable.

Given an operator K , we introduce the following notation

$$\tilde{K}(\varphi, v) := (\varphi, K_\varphi(v)),$$

where $K_\varphi(v) = R_\varphi \circ K \circ R_{\varphi^{-1}}(v)$.

(a) We have $P_\varphi(v) = (A_\varphi(v))(D_\varphi(v))$. But

$$(\varphi, v) \mapsto D_\varphi(v)$$

is smooth since $D_\varphi(v) = v_x/\varphi_x$ and $H^q(\mathbb{S}^1)$ is a Banach algebra. Also $H^{q-s}(\mathbb{S}^1)$ is a Banach algebra because $q - r > 1/2$. Hence the fact that $P_\varphi(v) \in H^{q-r}(\mathbb{S}^1)$ and our assumption ensure that

$$\tilde{P} : \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1) \rightarrow \mathcal{D}^q(\mathbb{S}^1) \times H^{q-r}(\mathbb{S}^1),$$

is of class C^m .

(b) Since

$$D_{(\varphi, v)} \tilde{A}(\delta\varphi, \delta v) = \begin{pmatrix} \text{id} & 0 \\ * & A_\varphi \end{pmatrix}$$

is a bounded, linear, invertible operator from $H^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1)$ to $H^q(\mathbb{S}^1) \times H^{q-r}(\mathbb{S}^1)$, we conclude, using the inverse mapping theorem on Banach spaces, that

$$\tilde{A}^{-1} : \mathcal{D}^q(\mathbb{S}^1) \times H^{q-r}(\mathbb{S}^1) \rightarrow \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1)$$

is of class C^m .

(c) Taking $P = A$ and $\delta\varphi_1 = v = u \circ \varphi$ in Lemma 26 when φ, v are smooth, we get

$$(17) \quad \partial_\varphi A_\varphi(v, v) = \{[u, A] \circ D\}_\varphi(u \circ \varphi) = -Q_\varphi(v).$$

Due to the density of the embedding $C^\infty(\mathbb{S}^1) \hookrightarrow H^q(\mathbb{S}^1)$, this relation is still valid for $\varphi \in \mathcal{D}^q(\mathbb{S}^1)$ and $v \in H^q(\mathbb{S}^1)$ and therefore

$$\tilde{Q} : \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1) \rightarrow \mathcal{D}^q(\mathbb{S}^1) \times H^{q-r}(\mathbb{S}^1),$$

is of class C^{m-1} . The assertion now follows from the chain rule. \square

Corollary 13. (*Smoothness of the H^s metric and its spray*) Let $s \geq 1/2$ be given and assume that $q \in (\frac{3}{2} + r, \infty) \cup \{1 + r\}$. Then the right-invariant, weak Riemannian metric defined on $\text{Diff}^\infty(\mathbb{S}^1)$ by the inertia operator $A = \Lambda^{2s}$ extends to a smooth weak Riemannian metric on the Banach manifold $\mathcal{D}^q(\mathbb{S}^1)$ with a smooth geodesic spray.

Proof. The smoothness of the metric results at once from Corollary 2 and formula (8). The smoothness of the spray (12) is a consequence of Proposition 12. \square

For further purpose, we will also state the following interesting property of the spray.

Lemma 14. *Under the hypotheses of Proposition 12 the mapping*

$$(\varphi, v) \mapsto S_\varphi(v)$$

is bounded in $H^q(\mathbb{S}^1)$ on bounded subsets of $\mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1)$.

Proof. Let first $B \subset H^q(\mathbb{S}^1)$ be bounded. In addition, we use the notation $P(u) := (Au)u_x$, $Q(u) := [A, u]u_x$, and $S(u) := P(u) + Q(u)$, introduced in Proposition 12. Since $H^{q-r}(\mathbb{S}^1)$ is a Banach algebra, we conclude that

$$\|P(u)\|_{H^{q-r}} = \|(Au)u_x\|_{H^{q-r}} \leq \|Au\|_{H^{q-r}} \|u_x\|_{H^{q-r}}.$$

Observing that $A \in \mathcal{L}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$ and using the fact that $r \geq 1$, we find a positive constant M_1 such that $\|P(u)\|_{H^{q-r}} \leq M_1$ for all $u \in B$. By our assumption on $A_\varphi(v)$ relation (17) holds true, from which we conclude at $\varphi = id$ that

$$Q = \partial_\varphi A_\varphi|_{\varphi=id} \in \mathcal{L}^2(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1)).$$

Thus there is a positive constant M_2 such that $\|Q(u)\|_{H^{q-r}} \leq M_2$ for all $u \in B$. Combining this with the fact that $A^{-1} \in \mathcal{L}(H^{q-r}(\mathbb{S}^1), H^q(\mathbb{S}^1))$, we find an $M > 0$ such that

$$\|S(u)\|_{H^{q-r}} \leq M \quad \text{for all } u \in B.$$

The assertion is now a consequence of (41) and Lemma 36, since $S_\varphi(v) = (R_\varphi \circ S \circ R_{\varphi^{-1}})(v)$. \square

6. EXISTENCE RESULTS FOR GEODESICS ON $\text{Diff}^\infty(\mathbb{S}^1)$

In this section, we will prove local existence and uniqueness of the initial value problem for the geodesics of the right-invariant H^s metric (and more generally for any right-invariant weak Riemannian metric for which the inertia operator satisfies the hypothesis of Theorem 1 on the Fréchet-Lie group $\text{Diff}^\infty(\mathbb{S}^1)$). For this we shall use the Banach approximation $\{\mathcal{D}^q(\mathbb{S}^1)\}_{q>3/2}$ of $\text{Diff}^\infty(\mathbb{S}^1)$ and the corresponding results of the previous section. Here the remarkable observation that the maximal interval of existence is independent of the parameter q (cf. Lemma 15) in the approximation

$$\text{Diff}^\infty(\mathbb{S}^1) = \bigcap_q \mathcal{D}^q(\mathbb{S}^1)$$

is most helpful. It is the essential ingredient which makes it possible to avoid Nash-Moser type schemes but to use a simple direct argument, cf. [12, Lemma 8] to prove the main result Theorem 17. We note that Lemma 15 is inspired by [11, Theorem 12.1].

In what follows, we start with a right-invariant metric on $\text{Diff}^\infty(\mathbb{S}^1)$ and we assume that its inertia operator A is a Fourier multiplier of order $r \geq 1$. We suppose further that, for all $q \in (\frac{3}{2} + r, \infty) \cup \{1 + r\}$, A induces an isomorphism from $H^q(\mathbb{S}^1)$ onto $H^{q-r}(\mathbb{S}^1)$ and that the mapping

$$(18) \quad (\varphi, v) \mapsto A_\varphi(v) = R_\varphi \circ A \circ R_{\varphi^{-1}}(v), \quad T\mathcal{D}^q(\mathbb{S}^1) \rightarrow H^{q-r}(\mathbb{S}^1)$$

is smooth. Under these assumptions Proposition 12 and Lemma 14 are available for any $n \in \mathbb{N}$. We recall the notation introduced in the previous sections to describe the spray in a local chart:

$$S(u) := A^{-1} \{[A, u]u_x - 2(Au)u_x\}, \quad u \in T_{id}\mathcal{D}^q(\mathbb{S}^1) \simeq H^q(\mathbb{S}^1)$$

and

$$S_\varphi(v) := R_\varphi \circ S \circ R_{\varphi^{-1}}(v), \quad \varphi \in \mathcal{D}^q(\mathbb{S}^1), v \in T_\varphi \mathcal{D}^q(\mathbb{S}^1) \simeq H^q(\mathbb{S}^1).$$

By Proposition 12 and the Picard-Lindelöf theorem there is, given any $(\varphi_0, v_0) \in T\mathcal{D}^q(\mathbb{S}^1)$, a local solution (φ, v) of

$$(19) \quad \begin{cases} \varphi_x = v, & \varphi(0) = \varphi_0, \\ v' = S_\varphi(v), & v(0) = v_0. \end{cases}$$

This solution can be continued to a *unique non-extendable* solution

$$(\varphi, v) \in C^\infty(J_q(\varphi_0, v_0), T\mathcal{D}^q(\mathbb{S}^1))$$

on the *maximal interval of existence*

$$J_q(\varphi_0, v_0) = (t^-(\varphi_0, v_0), t_q^+(\varphi_0, v_0)),$$

where $t_q^-(\varphi_0, v_0) < 0$ and $t_q^+(\varphi_0, v_0) > 0$. Moreover, it follows from Theorem 7.6 in [1] and Lemma 14 that if $t^-(\varphi_0, v_0) > -\infty$ or $t_q^+(\varphi_0, v_0) < \infty$ then

$$\lim_{t \downarrow t_q^-} \|(\varphi(t), v(t))\|_{H^q \times H^q} + \lim_{t \uparrow t_q^+} \|(\varphi(t), v(t))\|_{H^q \times H^q} = \infty.$$

Furthermore, letting

$$dom^{(q)} := \bigcup_{(\varphi_0, v_0) \in T\mathcal{D}^q(\mathbb{S}^1)} J_q(\varphi_0, v_0) \times \{(\varphi_0, v_0)\}$$

and

$$\Phi_q(t, (\varphi_0, v_0)) := (\varphi(t), v(t)), \quad t \in J_q(\varphi_0, v_0),$$

we know that $dom^{(q)}$ is open in $\mathbb{R} \times T\mathcal{D}^q(\mathbb{S}^1)$ and that

$$(20) \quad \Phi_q \in C^\infty(dom^{(q)}, \mathbb{R} \times T\mathcal{D}^q(\mathbb{S}^1)),$$

cf. Section 10 in [1]. The mapping Φ_q is called the *flow* on $T\mathcal{D}^q(\mathbb{S}^1)$, induced by the vector field $(v, S_\varphi(v))$ and $dom^{(q)}$ is its *maximal domain of definition*.

Our aim is to prove well-posedness of the Cauchy problem (19) on the smooth manifold $T\text{Diff}^\infty(\mathbb{S}^1)$. In order to do so, we aim to use a Banach manifold approximation of $T\text{Diff}^\infty(\mathbb{S}^1)$ based on the fact that

$$\bigcap_{q \geq 0} H^q(\mathbb{S}^1) = C^\infty(\mathbb{S}^1).$$

To follow this approach we need precise regularity properties of solutions to (19) on each level $H^q(\mathbb{S}^1)$ with $q > (3/2) + r$. More precisely, assume that $(\varphi_0, v_0) \in T\mathcal{D}^{q+1}(\mathbb{S}^1)$. Then we may solve (19) in $T\mathcal{D}^q(\mathbb{S}^1)$ and in $T\mathcal{D}^{q+1}(\mathbb{S}^1)$. Since solutions on each level are non-extendable, we clearly have

$$(21) \quad J_{q+1}(\varphi_0, v_0) \subset J_q(\varphi_0, v_0).$$

To rule out the possibility that $J_{q+1}(\varphi_0, v_0)$ is a proper subset of $J_q(\varphi_0, v_0)$, which could lead to $\bigcap_q J_q(\varphi_0, v_0) = \{0\}$, we need the following auxiliary considerations. Given $\sigma \geq 0$, let $\{T(s); s \in \mathbb{R}\}$ denote the translation group in $H^\sigma(\mathbb{S}^1)$, i.e.

$$T(s)v(x) := v(x+s), \quad v \in H^\sigma(\mathbb{S}^1), \quad x \in \mathbb{S}^1.$$

It is known that $\{T(s); s \in \mathbb{R}\}$ is a strongly continuous group in $\mathcal{L}(H^\sigma(\mathbb{S}^1))$. Its infinitesimal generator is given by ∂_x with domain of definition $H^{\sigma+1}(\mathbb{S}^1)$. This means in particular that, given $v_0 \in H^{\sigma+1}(\mathbb{S}^1)$, we have that

$$[s \mapsto T(s)v_0] \in C^1(\mathbb{R}, H^\sigma(\mathbb{S}^1))$$

with

$$\frac{d}{ds}T(s)v_0 = T(s)(\partial_x v_0), \quad s \in \mathbb{R}.$$

There is also a one parameter group of right translations in $\mathcal{D}^q(\mathbb{S}^1)$ for which we use the same notation $T(s)\varphi_0(x) := \varphi_0(x+s)$ for $\varphi_0 \in \mathcal{D}^\sigma(\mathbb{S}^1)$ and $x \in \mathbb{S}^1$.

Finally, in the following auxiliary result, we resort on the notation, introduced in (21).

Lemma 15. *Given $(\varphi_0, v_0) \in T\mathcal{D}^{q+1}(\mathbb{S}^1)$, we have $J_{q+1}(\varphi_0, v_0) = J_q(\varphi_0, v_0)$.*

Proof. Let

$$\Phi_q(\cdot, (\varphi_0, v_0)) = (\varphi, v) \in C^\infty(J_q(\varphi_0, v_0), T\mathcal{D}^q(\mathbb{S}^1))$$

be the solution to (19) with initial datum (φ_0, v_0) . From (20) we easily deduce that

$$(22) \quad \frac{d}{ds}\Phi_q(t, T(s)(\varphi_0, v_0))|_{s=0} = D_{(\varphi, v)}\Phi_q(t, (\varphi_0, v_0))(\varphi_{0,x}, v_{0,x}).$$

On the other hand, the spray $(\varphi, S_\varphi(v))$ is $\mathcal{D}^q(\mathbb{S}^1)$ -equivariant. This implies particularly that

$$\Phi_q(t, T(s)(\varphi_0, v_0)) = T(s)\Phi_q(t, (\varphi_0, v_0)) \quad \text{for all } t \in J_q(\varphi_0, v_0), s \in \mathbb{R}.$$

Thus the left hand side of (22) equals

$$\frac{d}{ds}T(s)\Phi_q(t, (\varphi_0, v_0))|_{s=0} = \partial_x \Phi_q(t, (\varphi_0, v_0)) = (\varphi_x(t), v_x(t)).$$

Combining these observations, we get

$$D_{(\varphi, v)}\Phi_q(t, (\varphi_0, v_0))(\varphi_{0,x}, v_{0,x}) = (\varphi_x(t), v_x(t)).$$

But (20) reveals that the left hand side of the latter identity belongs to $H^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1)$, which in turn implies that

$$(\varphi(t), v(t)) \in T\mathcal{D}^{q+1}(\mathbb{S}^1) \quad \text{for all } t \in J_q(\varphi_0, v_0).$$

By the unique solvability of (19), we conclude that

$$J_q(\varphi_0, v_0) \subset J_{q+1}(\varphi_0, v_0).$$

Invoking (21), the proof is completed. \square

Remark 16. Lemma 15 states that there is no loss of spatial regularity during the evolution of (19). By reversing the time direction, it follows from the unique solvability that there is also no gain of regularity in the following sense: Let $(\varphi_0, v_0) \in T\mathcal{D}^q(\mathbb{S}^1)$ be given and assume that $(\varphi(t_1), v(t_1)) \in T\mathcal{D}^{q+1}(\mathbb{S}^1)$ for some $t_1 \in J_q(\varphi_0, v_0)$. Then $(\varphi_0, v_0) \in T\mathcal{D}^{q+1}(\mathbb{S}^1)$.

Theorem 17. *Let (18) be satisfied and consider the geodesic flow on the tangent bundle $T\text{Diff}^\infty(\mathbb{S}^1)$ induced by the inertia operator A . Then, given any $(\varphi_0, v_0) \in T\text{Diff}^\infty(\mathbb{S}^1)$, there exists a unique non-extendable solution*

$$(\varphi, v) \in C^\infty(J, T\text{Diff}^\infty(\mathbb{S}^1))$$

of (19) on the maximal interval of existence $J = (t^-, t^+)$. If either $t^- > -\infty$ or $t^+ < \infty$ then

$$\lim_{t \downarrow t^-} \|(\varphi(t), v(t))\|_{H^{1+r} \times H^{1+r}} + \lim_{t \uparrow t^+} \|(\varphi(t), v(t))\|_{H^{1+r} \times H^{1+r}} = \infty.$$

Proof. The result follows from (20), Lemma 15 and [12, Lemma 8], cf. the proof of Theorem 12 in [12]. \square

Corollary 18. *Let $s \geq 1/2$ be given and consider the right-invariant Sobolev H^s -metric on $\text{Diff}^\infty(\mathbb{S}^1)$. Then the corresponding geodesic equation has for any initial data in the tangent bundle $T\text{Diff}^\infty(\mathbb{S}^1)$ a unique non-extendable smooth solution $(\varphi, v) \in C^\infty(J, T\text{Diff}^\infty(\mathbb{S}^1))$. The maximal interval of existence J is open and contains 0. Moreover, if $J = (t^-, t^+)$ and either $t^- > -\infty$ or $t^+ < \infty$ then (φ, v) blows up in $H^{1+2s}(\mathbb{S}^1)$, i.e.*

$$\lim_{t \downarrow t^-} \|(\varphi(t), v(t))\|_{H^{1+2s} \times H^{1+2s}} + \lim_{t \uparrow t^+} \|(\varphi(t), v(t))\|_{H^{1+2s} \times H^{1+2s}} = \infty.$$

Proof. Let $s \geq 1/2$ be given. Then Corollary 2 ensures that the smoothness assertion in (18) is satisfied for $\mathbf{op}((1+k^2)^s)$. The hypothesis on the isomorphy in (18) is obvious in this case. Thus the result follows from Theorem 17. \square

It is worth emphasizing that the blow up result in Corollary 18 only represents a necessary condition. For particular options of the inertia operator A it is known that the blow up occurs in weaker norms than $H^{1+2s}(\mathbb{S}^1)$. Indeed, for $A = \mathbf{op}((1+k^2))$, which leads to the periodic Camass-Holm equation, cf. [7], the precise blow up scenario is known: a classical solution u blows up in finite time if and only if

$$\lim_{t \rightarrow t^+} \left(\min_{x \in \mathbb{S}^1} \{u_x(t, x)\} \right) = -\infty,$$

cf. [5], which is considerably weaker than blow up in $H^3(\mathbb{S}^1)$.

On the other hand there are several evolution equations, different from the Camassa-Holm equation, e.g. the Constantin-Lax-Majda equation, briefly discussed in Section 8, for which the blow up mechanism is much less understood and so far no sharper results than blow up in $H^{1+2s}(\mathbb{S}^1)$ seems to be known.

7. EXPONENTIAL MAPPING AND GEODESIC DISTANCE ON $\text{Diff}^\infty(\mathbb{S}^1)$

The geodesic flow of a smooth spray on a Banach manifold M satisfies the following remarkable property

$$\varphi(t, x_0, su_0) = \varphi(st, x_0, u_0),$$

which is a consequence of the quadratic nature of the geodesic equation [22]. Therefore, the *exponential mapping* \exp_{x_0} , defined as the time one of the flow is well defined in a neighbourhood of 0 in $T_{x_0}M$ for each point x_0 . It

is moreover a local diffeomorphism from a neighbourhood V of 0 in $T_{x_0}M$ onto a neighbourhood $U(x_0)$ of x_0 in M [22]. This last assertion may be false on a *Fréchet manifold* and in particular on $\text{Diff}^\infty(\mathbb{S}^1)$. One may find useful to recall on this occasion that the *group exponential* of $\text{Diff}^\infty(\mathbb{S}^1)$ is not a local diffeomorphism [25]. Moreover, the Riemannian exponential map for the L^2 metric (Burgers equation) on $\text{Diff}^\infty(\mathbb{S}^1)$ is not a local C^1 -diffeomorphism near the origin [6]. Nevertheless, it has been established in [6], that for the Camassa-Holm equation – which corresponds to the Euler equation of the H^1 metric on $\text{Diff}^\infty(\mathbb{S}^1)$ – and more generally for H^k metrics ($k \geq 1$) (see [7]), the Riemannian exponential map was in fact a smooth local diffeomorphism. This result is still true for H^s right-invariant metrics on $\text{Diff}^\infty(\mathbb{S}^1)$ provided $s \in [1/2, +\infty)$. The proof of Theorem 4 is similar to the one given in [12] and will be omitted. It requires only the smoothness of the spray on $T\mathcal{D}^q(\mathbb{S}^1)$ for all q large enough.

We will finish this section by a remark concerning the geodesic semi-distance d_s induced by the H^s metric and defined as the greatest lower bound of path-lengths $L_s(\varphi)$, for piecewise C^1 paths $\varphi(t)$ in $\text{Diff}^\infty(\mathbb{S}^1)$ joining φ_0 and φ_1 . It was first shown in [24], that this semi-distance vanishes identically for the L^2 right-invariant metric on the diffeomorphisms group of any compact manifold. More recently, it was shown in [3] that d_s vanishes identically on $\text{Diff}^\infty(\mathbb{S}^1)$ if $s \in [0, 1/2]$, whereas d_s is a distance for $s > 1/2$

$$\forall \varphi_0, \varphi_1 \in \text{Diff}^\infty(\mathbb{S}^1), \quad \varphi_0 \neq \varphi_1 \Rightarrow d_s(\varphi_0, \varphi_1) > 0.$$

Since the geodesic spray of the weak H^s right-invariant Riemannian metric on $\mathcal{D}^q(\mathbb{S}^1)$ is smooth for $q \geq 1 + 2s$ and $s \geq 1/2$, its exponential mapping on $\mathcal{D}^q(\mathbb{S}^1)$ is a diffeomorphism from a neighbourhood V of 0 in $H^q(\mathbb{S}^1)$ to a neighbourhood U of the identity in $\mathcal{D}^q(\mathbb{S}^1)$. This leads to the existence of *local polar coordinates* in the *normal chart* U . These coordinates are defined as follows. Given $\varphi \in U - \{\text{id}\}$, there is a $v \in V \setminus \{0\}$ such that $\varphi = \text{exp}(v)$. Letting now

$$w := v / \|v\|_{H^s}, \quad \rho := \|v\|_{H^s},$$

we have that $\varphi = \text{exp}(\rho w)$ and (ρ, w) are called the *polar coordinates* of $\varphi \in U - \{\text{id}\}$. Notice that (ρ, w) depend smoothly of φ and that $\rho(\varphi) \rightarrow 0$ as $\varphi \rightarrow \text{id}$.

As can be checked in [22], the following result is valid not only for a strong Riemannian metric but also for a *weak* Riemannian metric, *provided* there exists a compatible, symmetric covariant derivative.

Lemma 19. *For a piecewise C^1 curve $\gamma : [a, b] \rightarrow U(\varphi_0) - \{\varphi_0\}$, we have the inequality*

$$L(\gamma) \geq |\rho(b) - \rho(a)|.$$

However, it should be noticed that Lemma 19 does not imply that the geodesic semi-distance is in fact a distance. What Lemma 19 says, is that the length of any path which *lies inside* the normal neighbourhood is bounded below by $r := |\rho(b) - \rho(a)|$. However for a path which leaves the normal neighbourhood, this might not be true. Such a path could leave the normal neighbourhood *before* leaving the (weak ball) of radius r defined as

$$B_s(\text{id}, r) := \{\varphi \in U; \rho(\varphi) \leq r\}.$$

In fact this happens for the critical exponent $s = 1/2$ as it follows from [3].

8. EULER EQUATIONS ON THE HOMOGENEOUS SPACE $\text{Diff}^\infty(\mathbb{S}^1)/\text{Rot}(\mathbb{S}^1)$

In this section, we will apply our main theorems to some geodesic equations on the homogeneous space $\text{Diff}^\infty(\mathbb{S}^1)/\text{Rot}(\mathbb{S}^1)$. Since the proof are very similar to what has been done so far, we will not give all the details but only point out new difficulties that may arise.

Let $\text{Rot}(\mathbb{S}^1)$ denote the subgroup of all rigid rotations of \mathbb{S}^1 and

$$\text{Diff}^\infty(\mathbb{S}^1)/\text{Rot}(\mathbb{S}^1)$$

be the corresponding homogeneous space of right cosets. It is not difficult to verify that the restriction of the canonical projection

$$\text{Diff}^\infty(\mathbb{S}^1) \rightarrow \text{Diff}^\infty(\mathbb{S}^1)/\text{Rot}(\mathbb{S}^1)$$

to the subgroup $\text{Diff}_1^\infty(\mathbb{S}^1)$ of $\text{Diff}^\infty(\mathbb{S}^1)$ consisting of all diffeomorphisms of $\mathbb{S}^1 \simeq \mathbb{R}/\mathbb{Z}$ which fixes one arbitrarily point (which we can take to be 0) is a diffeomorphism.

Remark 20. Since $\text{Diff}^\infty(\mathbb{S}^1)$ is simple: it possesses no nontrivial normal subgroups, cf. [15], the above identification $\text{Diff}_1^\infty(\mathbb{S}^1)$ with $\text{Diff}^\infty(\mathbb{S}^1)/\text{Rot}(\mathbb{S}^1)$ is possible but the restriction of the canonical projection to $\text{Diff}_1^\infty(\mathbb{S}^1)$ is not a group morphism.

The Lie algebras of $\text{Diff}_1^\infty(\mathbb{S}^1)$ and $\text{Rot}(\mathbb{S}^1)$ are given respectively by

$$\mathcal{C}_0^\infty(\mathbb{S}) := \{u \in C^\infty(\mathbb{S}^1); u_x(0) = 0\} \quad \text{and} \quad \mathbb{R} \cdot w_0,$$

where w_0 stands for the constant function with value 1.

Let $A = \mathbf{op}(p(k))$ be a L^2 -symmetric⁵ Fourier multiplier on $C^\infty(\mathbb{S}^1)$ and assume that its symbol satisfies

$$(23) \quad p(k) = 0 \quad \text{iff } k = 0.$$

Then $\ker A = \mathbb{R} \cdot w_0$. Furthermore $\text{ad}_{w_0} = -D$ and $\text{ad}_{w_0}^* = -D$. Thus hypothesis (ii) of Proposition 9 is satisfied. Therefore, A induces a right-invariant Riemannian metric on $\text{Diff}_1^\infty(\mathbb{S}^1)$ and the corresponding Euler equation is given by

$$(24) \quad u_t = -A^{-1} \{(Au)_x u + 2(Au)u_x\}.$$

To solve this evolution equation, we also need a suitable Banach space approximation of $\text{Diff}_1^\infty(\mathbb{S}^1)$. For this fix an arbitrary point $x_0 \in \mathbb{S}^1$ and set

$$\mathcal{D}_1^q(\mathbb{S}^1) := \{\varphi \in \mathcal{D}^q(\mathbb{S}^1); \varphi(x_0) = x_0\} \quad \text{for } q > 3/2.$$

Then $\mathcal{D}_1^q(\mathbb{S}^1)$ is a Banach manifold modeled on the Banach space

$$H_0^q(\mathbb{S}^1) := \{u \in H^q(\mathbb{S}^1); u(x_0) = 0\}.$$

and a topological group. Let

$$\hat{H}_0^\sigma(\mathbb{S}^1) := \{m \in H^\sigma(\mathbb{S}^1); \hat{m}(0) = 0\}$$

⁵which is equivalent to the fact that its symbol p is real.

for $\sigma \geq 0$. If $A = \mathbf{op}(p(k))$ is a Fourier multiplier of order $r \geq 1$, satisfying (23), then A extends to $H_0^q(\mathbb{S}^1)$ such that

$$(25) \quad A \in \text{Isom}(H_0^q(\mathbb{S}^1), \hat{H}_0^{q-r}(\mathbb{S}^1)),$$

for all $q \in (3/2 + r, \infty) \cup \{r + 1\}$. From this we conclude that the map

$$(26) \quad \mathcal{D}_1^q(\mathbb{S}^1) \times H_0^q(\mathbb{S}^1) \rightarrow H_0^q(\mathbb{S}^1), \quad (\varphi, v) \mapsto S_\varphi(v) := R_\varphi \circ S \circ R_{\varphi^{-1}}(v),$$

where $S(u) := A^{-1}\{[A, u]u_x - 2(Au)u_x\}$, is well defined. Furthermore, if

$$(\varphi, v) \mapsto A_\varphi(v) = R_\varphi \circ A \circ R_{\varphi^{-1}}(v).$$

is of class C^m from $\mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1)$ to $H^{q-r}(\mathbb{S}^1)$ then its restriction to $\mathcal{D}_1^q(\mathbb{S}^1) \times H_0^q(\mathbb{S}^1)$ is also of class C^m .

Theorem 21. *Let $A = \mathbf{op}(p(k))$ be a Fourier multiplier of order $r \geq 1$ with a real symbol p , satisfying*

$$p(k) = 0 \iff k = 0.$$

Assume that in addition that

$$(\varphi, v) \mapsto A_\varphi(v) = R_\varphi \circ A \circ R_{\varphi^{-1}}(v).$$

is smooth from $T\mathcal{D}^q(\mathbb{S}^1)$ to $H^{q-r}(\mathbb{S}^1)$. Then, given any $(\varphi_0, v_0) \in T\text{Diff}_1^\infty(\mathbb{S}^1)$, there exists a unique non-extendable solution

$$(\varphi, v) \in C^\infty(J, T\text{Diff}_1^\infty(\mathbb{S}^1))$$

of the Cauchy problem for the associated geodesic spray

$$\begin{cases} \varphi_x = v, & \varphi(0) = \varphi_0 \\ v' = S_\varphi(v), & v(0) = v_0. \end{cases}$$

on the maximal interval of existence $J = (t^-, t^+)$. If either $t^- > -\infty$ or $t^+ < \infty$ then

$$\lim_{t \downarrow t^-} \|(\varphi(t), v(t))\|_{H^{1+r} \times H^{1+r}} + \lim_{t \uparrow t^+} \|(\varphi(t), v(t))\|_{H^{1+r} \times H^{1+r}} = \infty.$$

Sketch of proof. The proof requires Proposition 12 but the argument there has to be slightly modified to work in the setting here. Indeed, given $\varphi \in \mathcal{D}_1^q(\mathbb{S}^1)$, A_φ takes values in the vector space

$$R_\varphi(\hat{H}_0^{q-r}(\mathbb{S}^1)) = \left\{ m \in H^\sigma(\mathbb{S}^1); (\widehat{m \circ \varphi^{-1}})(0) = 0 \right\}$$

which depends on φ , which makes it difficult to apply the inverse mapping theorem in this setting as it is required in part (b) of the proof of Proposition 12. To overcome this difficulty, we replace the operator

$$\tilde{A}(\varphi, v) := (\varphi, A_\varphi(v)),$$

in the proof of Proposition 12 by

$$\tilde{A}(\varphi, v) := (\varphi, \varphi_x A_\varphi(v)),$$

and we notice that $m \mapsto \varphi_x m$ is a linear isomorphism from $R_\varphi(\hat{H}_0^{q-r}(\mathbb{S}^1))$ onto $\hat{H}_0^{q-r}(\mathbb{S}^1)$. \square

We briefly discuss two special options of inertia operators A , namely $A = \mathbf{op}(k^2)$ and $A = \mathbf{op}(|k|)$. In the first case $A = \mathbf{op}(k^2)$ the Euler equation reads as

$$(27) \quad u_{txx} = -2u_x u_{xx} - uu_{xxx}, \quad t > 0, \quad x \in \mathbb{S}^1,$$

and is known as the periodic *Hunter-Saxton* equation, cf. [18, 33, 4, 23].

For the inertia operator $A = \mathbf{op}(|k|)$ we get the so called CLM equation, cf. [8, 32, 13].

$$(28) \quad \partial_t(Hu) + uHu_{xx} + 2u_x Hu_x = 0, \quad t > 0, \quad x \in \mathbb{S}^1,$$

where $H = \mathbf{op}(i \operatorname{sgn}(k))$ denotes the Hilbert transform, acting on the spatial variable $x \in \mathbb{S}^1$. Note that $\mathbf{op}(|k|) = H \circ D$.

Clearly, both symbols $(k^2)_{k \in \mathbb{Z}}$ and $(|k|)_{k \in \mathbb{Z}}$ satisfy (23). Moreover they also fulfill the hypotheses of Theorem 1, so that Theorem 21 is applicable to (27) and (28).

Remark 22. To conclude this section, it could be worth to bring together the present work with the right-invariant metric defined by the inertia operator

$$A := HD(D^2 + 1)$$

defined on the diffeomorphism group of the circle which fixes the three points $-1, 0, 1$. This metric has been related with the Weil-Petersson metric on the universal Teichmüller space $T(1)$ in [28]. The corresponding geodesic flow has been extensively studied in [14]. Recall first that $\mathcal{D}^s(\mathbb{S}^1)$, the space of homeomorphisms of class H^s as well as their inverse is a topological group only for $s > 3/2$ and that $3/2$ is therefore a critical exponent. One of the main results in [14] is that, the inertia operator A defines on a suitable replacement for the “ $H^{3/2}$ diffeomorphism group”, a right-invariant *strong Riemannian structure* which is moreover geodesically complete (i.e., geodesics are defined for all times).

Our point of view in this paper is completely different in the sense that we work on a well defined topological group $\mathcal{D}^s(\mathbb{S}^1)$ for $s > 3/2$ equipped with a Banach manifold structure. The price to pay for this nice structure is the fact that the metric only defines a *weak Riemannian structure*. Nevertheless, we have been able to show local existence of the geodesics, also in this context.

APPENDIX A. FOURIER MULTIPLIERS

Here and in the following we use the notation

$$\mathbf{e}_n(x) = \exp(2\pi i n x),$$

for $n \in \mathbb{Z}$ and $x \in \mathbb{S}^1$.

Lemma 23. *Let P a continuous linear operator on the Fréchet space $C^\infty(\mathbb{S}^1)$. Then the following three conditions are equivalent:*

- (1) P commutes with all rotations R_s .
- (2) $[P, D] = 0$, where $D = d/dx$.
- (3) For each $n \in \mathbb{Z}$, there is a $p(n) \in \mathbb{C}$ such that $P\mathbf{e}_n = p(n)\mathbf{e}_n$.

In that case, we say that P is a Fourier multiplier.

Since every smooth function on the unit circle \mathbb{S}^1 can be represented by its Fourier series, we get that

$$(29) \quad (Pu)(x) = \sum_{k \in \mathbb{Z}} p(k) \hat{u}(k) \mathbf{e}_k(x),$$

for every Fourier multiplier P and every $u \in C^\infty(\mathbb{S}^1)$, where

$$\hat{u}(k) := \int_{\mathbb{S}^1} u(x) \mathbf{e}_{-k}(x) dx,$$

stands for the k -th Fourier coefficients of u . The sequence $p : \mathbb{Z} \rightarrow \mathbb{C}$ is called the *symbol* of P . We use also the notation $P := \mathbf{op}(p(k))$ for the Fourier multiplier induced by the sequence p .

Proof. Given $s \in \mathbb{R}$ and $u \in C^\infty(\mathbb{S}^1)$, let $u_s(x) := u(x + s)$. If P commutes with translations we have

$$(Pu)_s(x) = (Pu_s)(x).$$

Taking the derivative of both sides of this equation with respect to s at 0 and using the continuity of P , we get $DPu = PDu$ which proves the implication (1) \Rightarrow (2).

If $[P, D] = 0$, then both $P\mathbf{e}_n$ and \mathbf{e}_n are solutions of the linear differential equation

$$u' = (-2\pi i n)u$$

and are therefore equal up to a multiplicative constant $p(n)$. This proves that (2) \Rightarrow (3).

If $P\mathbf{e}_n = p(n)\mathbf{e}_n$, for each $n \in \mathbb{Z}$ and P is continuous, then we have representation (29). Therefore

$$\begin{aligned} (Pu)_s(x) &= \sum_{k \in \mathbb{Z}} p(k) \hat{u}(k) \mathbf{e}_k(x + s) \\ &= \sum_{k \in \mathbb{Z}} p(k) \hat{u}_s(k) \mathbf{e}_k(x) = (Pu_s)(x), \end{aligned}$$

which proves that (3) \Rightarrow (1). \square

Remark 24. Notice that the space of Fourier multipliers is a *commutative subalgebra* of the algebra of linear operators on $C^\infty(\mathbb{S}^1)$ which contains all linear differential operators with constant coefficients. Notice also that each Fourier multiplier defines an L^2 -symmetric operator

Remark 25. Notice that a Fourier multiplier P is L^2 -symmetric iff its symbol p is real.

A Fourier multiplier $P = \mathbf{op}(p(k))$ with symbol p is said to be of *order* $r \in \mathbb{R}$ if there exists a constant $C > 0$ such that

$$|p(k)| \leq C (1 + k^2)^{r/2},$$

for every $k \in \mathbb{Z}$. In that case, for each $q \geq r$, the operator P extends⁶ to a bounded linear operator from $H^q(\mathbb{S}^1)$ into $H^{q-r}(\mathbb{S}^1)$. We express this fact by the notation $P \in \mathcal{L}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$.

⁶Throughout this paper we consider Fourier operators of order $r \geq 1$, since our main results Theorem 1 is only true for this class of operators. It is nevertheless worth to mention that several results in this section remain true for operators of any positive order.

Let $(\varphi, v) \mapsto P_\varphi(v)$ be a smooth mapping on the Fréchet manifold $\text{Diff}^\infty(\mathbb{S}^1) \times C^\infty(\mathbb{S}^1)$, where P is linear in v . The partial Gâteaux derivative of P in the first variable φ and in the direction $\delta\varphi_1 \in C^\infty(\mathbb{S}^1)$ is a smooth map which is linear both in v and $\delta\varphi_1$ and that we will denote by

$$(30) \quad \partial_\varphi P_\varphi(v, \delta\varphi_1).$$

Therefore, the partial Gâteaux derivative of P in the variable φ is a mapping of three independent variables : $\varphi, v, \delta\varphi_1$. The second partial derivative of P in directions $\delta\varphi_1, \delta\varphi_2 \in C^\infty(\mathbb{S}^1)$ is the partial Gâteaux derivative of (30) in the variable φ and in the direction $\delta\varphi_2$. We will denote it by

$$\partial_\varphi^2 P_\varphi(v, \delta\varphi_1, \delta\varphi_2).$$

It can be checked that this expression is symmetric in $\delta\varphi_1, \delta\varphi_2$ (see [16]). Inductively, we define this way the n -th partial derivative of P in directions $\delta\varphi_1, \dots, \delta\varphi_n$ and we write it as

$$\partial_\varphi^n P_\varphi(v, \delta\varphi_1, \dots, \delta\varphi_n).$$

The space of linear operators on a Fréchet space is a locally convex topological vector space, but in general is not a Fréchet space (see [16]). For this reason, we will avoid taking limits and derivatives of linear operators. In the sequel, if such equalities appear for notational simplicity, it just means equality of mappings.

Let P denote a general Fourier multiplier on $C^\infty(\mathbb{S}^1)$. We will now study conjugation

$$P_\varphi := R_\varphi \circ P \circ R_{\varphi^{-1}}$$

of P with right translations R_φ , where $\varphi \in \text{Diff}^\infty(\mathbb{S}^1)$. We will derive a recursion formula for the n -th derivative with respect to φ of such operators. In addition, we provide a sufficient criterion on the symbol of the original operator P , which ensures that the n -th derivative $\partial_\varphi^n P_\varphi$ extends to an $(n+1)$ -linear mapping on suitable Sobolev spaces.

Lemma 26. *Let P be a continuous, linear operator on $C^\infty(\mathbb{S}^1)$ and let*

$$P_\varphi = R_\varphi P R_\varphi^{-1},$$

where $\varphi \in \text{Diff}^\infty(\mathbb{S}^1)$. Then, given $n \in \mathbb{N}$, we have

$$(31) \quad \partial_\varphi^n P_\varphi(v, \delta\varphi_1, \dots, \delta\varphi_n) = R_\varphi P_n(u_1, \dots, u_n) R_\varphi^{-1}(v),$$

where $u_i = \delta\varphi_i \circ \varphi^{-1}$ and P_n is the multilinear operator defined inductively by $P_0 = P$ and

$$(32) \quad P_{n+1}(u_1, \dots, u_{n+1}) = [u_{n+1} D, P_n(u_1, \dots, u_n)] - \sum_{i=1}^n P_n(u_1, \dots, u_i, x u_{n+1}, \dots, u_n).$$

Remark 27. For a Fourier multiplier, that is, if $[P, D] = 0$, we have

$$P_1(u_1) = [u_1, P] D,$$

and

$$P_2(u_1, u_2) = [u_1, [u_2, P]] D^2 + [u_1, P][u_2, D] D + [u_2, P][u_1, D] D.$$

Proof. Formula (31) is trivially true for $n = 0$. Now suppose it is true for some $n \in \mathbb{N}$, that is

$$\partial_\varphi^n P_\varphi(v, \delta\varphi_1, \dots, \delta\varphi_n) = R_\varphi P_n(u_1, \dots, u_n) R_\varphi^{-1}(v),$$

where $u_i = \delta\varphi_i \circ \varphi^{-1}$ for $1 \leq i \leq n$. Notice that, for fixed $\delta\varphi_1, \dots, \delta\varphi_n$

$$P_n(u_1, \dots, u_n) = P_n(\delta\varphi_1 \circ \varphi^{-1}, \dots, \delta\varphi_n \circ \varphi^{-1})$$

is a family of linear operator on $C^\infty(\mathbb{S}^1)$ indexed by φ and which depend on φ only through the u_i . Let $\varphi(s)$ be a smooth path in $\text{Diff}^\infty(\mathbb{S}^1)$ such that

$$\varphi(0) = \varphi, \quad \partial_s \varphi(s) \big|_{s=0} = \delta\varphi_{n+1}$$

and let $u_{n+1} = \delta\varphi_{n+1} \circ \varphi^{-1}$. We compute first

$$\dot{R}_\varphi := \partial_s R_{\varphi(s)} \big|_{s=0} = R_\varphi u_{n+1} D,$$

so that

$$R_\varphi^{-1} \dot{R}_\varphi = u_{n+1} D,$$

and

$$\dot{u}_i := \partial_s (\delta\varphi_i \circ \varphi(s)^{-1}) \big|_{s=0} = -u_{i,x} u_{n+1},$$

for $1 \leq i \leq n$. We have then

$$\dot{P}_n := \partial_s P_n(u_1, \dots, u_n) \big|_{s=0} = - \sum_{i=1}^n P_n(u_1, \dots, u_{i,x} u_{n+1}, \dots, u_n).$$

Finally, we have (simplifying the notation P_n for $P_n(u_1, \dots, u_n)$)

$$\begin{aligned} \partial_s R_\varphi P_n R_\varphi^{-1} \big|_{s=0} &= \dot{R}_\varphi P_n R_\varphi^{-1} + R_\varphi \dot{P}_n R_\varphi^{-1} - R_\varphi P_n (R_\varphi^{-1} \dot{R}_\varphi R_\varphi^{-1}) \\ &= R_\varphi (R_\varphi^{-1} \dot{R}_\varphi P_n - P_n R_\varphi^{-1} \dot{R}_\varphi) R_\varphi^{-1} + R_\varphi \dot{P}_n R_\varphi^{-1} \\ &= R_\varphi ([u_{n+1} D, P_n] + \dot{P}_n) R_\varphi^{-1}, \end{aligned}$$

which gives the recurrence relation (32), since

$$\partial_\varphi^{n+1} P_\varphi(v, \delta\varphi_1, \dots, \delta\varphi_{n+1}) = \partial_s (R_\varphi P_n(u_1, \dots, u_n) R_\varphi^{-1}(v)) \big|_{s=0},$$

the proof is complete. \square

Lemma 28. *Let P be a Fourier multiplier on $C^\infty(\mathbb{S}^1)$, and let P_n be the multilinear operator defined in Lemma 26 for some $n \in \mathbb{N}$. Then we have*

$$(33) \quad P_n(\mathbf{e}_{m_1}, \dots, \mathbf{e}_{m_n}) \mathbf{e}_{m_0} = p_n(m_0, m_1, \dots, m_n) \mathbf{e}_{m_0+m_1+\dots+m_n},$$

where the sequence p_n is defined inductively by $p_0 = p$ (the symbol of P) and

$$(34) \quad p_{n+1}(m_0, \dots, m_{n+1}) = (2\pi i) \left[(m_0 + \dots + m_n) p_n(m_0, \dots, m_n) - \sum_{j=0}^n m_j p_n(m_0, \dots, m_j + m_{n+1}, \dots, m_n) \right],$$

where $m_j \in \mathbb{Z}$ for $j = 1, \dots, n$.

Remark 29. For $n = 1$, we have

$$(35) \quad p_1(m_0, m_1) = (2\pi i)m_0(p_0(m_0) - p_0(m_0 + m_1))$$

and for $n = 2$, we get

$$(36) \quad p_2(m_0, m_1, m_2) = (2\pi i)^2 m_0 \left((m_0 + m_1 + m_2)p_0(m_0 + m_1 + m_2) \right. \\ \left. - (m_0 + m_1)p_0(m_0 + m_1) - (m_0 + m_2)p_0(m_0 + m_2) + m_0 p_0(m_0) \right).$$

Proof. Invoking lemma 23, the case $n = 0$ is clear. Suppose that equation (33) is true for some $n \geq 0$. Then, using recurrence relation (32), we have

$$\begin{aligned} P_{n+1}(\mathbf{e}_{m_1}, \dots, \mathbf{e}_{m_{n+1}})\mathbf{e}_{m_0} &= \mathbf{e}_{m_{n+1}} D(P_n(\mathbf{e}_{m_1}, \dots, \mathbf{e}_{m_n})\mathbf{e}_{m_0}) \\ &\quad - P_n(\mathbf{e}_{m_1}, \dots, \mathbf{e}_{m_n})(\mathbf{e}_{m_{n+1}} D\mathbf{e}_{m_0}) \\ &\quad - \sum_{j=1}^n P_n(\mathbf{e}_{m_1}, \dots, D\mathbf{e}_{m_j}\mathbf{e}_{m_{n+1}}, \dots, \mathbf{e}_{m_n}), \end{aligned}$$

which is equal to

$$\begin{aligned} (2\pi i) \left\{ (m_0 + \dots + m_n)p_n(m_0, \dots, m_n) - m_0 p_n(m_0 + m_{n+1}, \dots, m_n) \right. \\ \left. - \sum_{j=1}^n m_j p_n(m_0, \dots, m_j + m_{n+1}, \dots, m_n) \right\} \mathbf{e}_{m_0 + \dots + m_{n+1}}. \end{aligned}$$

This shows that equation (33) is true for $n + 1$ with

$$\begin{aligned} p_{n+1}(m_0, \dots, m_{n+1}) &= (2\pi i) \left[(m_0 + \dots + m_n)p_n(m_0, \dots, m_n) \right. \\ &\quad \left. - \sum_{j=0}^n m_j p_n(m_0, \dots, m_j + m_{n+1}, \dots, m_n) \right] \end{aligned}$$

and achieves the proof. \square

Corollary 30. *Under the notations of Lemma (28), we have*

$$(37) \quad p_n(m_0, m_1, \dots, m_n) = (2\pi i)^n m_0 \left[\sum_{p=0}^n (-1)^p \sum_{\substack{I \subset \{1, \dots, n\}, \\ |I|=p}} f_n(m_0 + \sum_{j \in I} m_j) \right],$$

for each $n \geq 1$, where $f_n(k) = k^{n-1}p_0(k)$, $k \in \mathbb{Z}$.

Proof. For $n = 1$, we have

$$p_1(m_0, m_1) = (2\pi i)m_0(p_0(m_0) - p_0(m_0 + m_1))$$

so equation (37) is true for $n = 1$. Now, suppose inductively that this equation is valid for some $n \geq 1$. Using the recurrence relation (34), we get

$$\begin{aligned} p_{n+1}(m_0, m_1, \dots, m_{n+1}) &= (2\pi i)^{n+1} m_0 \sum_{p=0}^n (-1)^p \sum_{\substack{I \subset \{1, \dots, n\}, \\ |I|=p}} \left\{ \right. \\ &\quad (m_0 + \dots + m_n) f_n(m_0 + \sum_{j \in I} m_j) \\ &\quad - \sum_{k=1}^n m_k f_n(m_0 + \sum_{j \in I} m_j + \delta_I(k) m_{n+1}) \\ &\quad \left. - (m_0 + m_{n+1}) f_n(m_0 + \sum_{j \in I} m_j + m_{n+1}) \right\}, \end{aligned}$$

which can be rewritten as

$$\begin{aligned} (2\pi i)^{n+1} m_0 \sum_{p=0}^n (-1)^p \sum_{\substack{I \subset \{1, \dots, n\}, \\ |I|=p}} \left\{ (m_0 + \sum_{j \in I} m_j) f_n(m_0 + \sum_{j \in I} m_j) \right. \\ \left. - (m_0 + \sum_{j \in I} m_j + m_{n+1}) f_n(m_0 + \sum_{j \in I} m_j + m_{n+1}) \right\}. \end{aligned}$$

using the fact that $f_{n+1}(t) = t f_n(t)$, we have therefore

$$\begin{aligned} p_{n+1}(m_0, m_1, \dots, m_{n+1}) &= (2\pi i)^{n+1} m_0 \sum_{p=0}^n (-1)^p \sum_{\substack{I \subset \{1, \dots, n\}, \\ |I|=p}} \left\{ \right. \\ &\quad f_{n+1}(m_0 + \sum_{j \in I} m_j) - f_{n+1}(m_0 + \sum_{j \in I} m_j + m_{n+1}) \left. \right\}, \end{aligned}$$

which is equal to

$$\begin{aligned} (2\pi i)^{n+1} m_0 \left\{ \sum_{p=0}^n (-1)^p \sum_{\substack{I \subset \{1, \dots, n+1\}, \\ |I|=p, n+1 \notin I}} f_{n+1}(m_0 + \sum_{j \in I} m_j) \right. \\ \left. + \sum_{p=0}^n (-1)^{p+1} \sum_{\substack{I \subset \{1, \dots, n+1\}, \\ |I|=p+1, n+1 \in I}} f_{n+1}(m_0 + \sum_{j \in I} m_j) \right\}. \end{aligned}$$

But this last expression is exactly

$$(2\pi i)^{n+1} m_0 \sum_{p=0}^{n+1} (-1)^p \sum_{\substack{I \subset \{1, \dots, n+1\}, \\ |I|=p}} f_{n+1}(m_0 + \sum_{j \in I} m_j),$$

which achieves the proof. \square

Lemma 31. *Let P be a Fourier multiplier of order $r \geq 1$ and $q \geq r+1$. Let P_n be the $(n+1)$ -multilinear operator defined by the recurrence relation (32)*

with $P_0 := P$. Suppose that there exists a constant $C_n > 0$, such that

$$(38) \quad |p_n(m_0, \dots, m_n)| \leq C_n(1 + m_0^2)^{r/2} \cdots (1 + m_n^2)^{r/2}$$

for all $m_j \in \mathbb{Z}$. Then P_n extends to a bounded multilinear operator

$$P_n \in \mathcal{L}^{n+1}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1)).$$

Proof. By virtue of Proposition 28, we have

$$\|P_n(u_1, \dots, u_n)u_0\|_{H^{q-r}}^2 = \sum_{l \in \mathbb{Z}} (1 + l^2)^{q-r} \left| \sum_{m_0 + \dots + m_n = l} \hat{u}_0(m_0) \cdots \hat{u}_n(m_n) p_n(m_0, \dots, m_n) \right|^2,$$

for any choice of smooth functions u_0, u_1, \dots, u_n , since $(\mathbf{e}_l)_{l \in \mathbb{Z}}$ is an orthogonal system in $H^{q-r}(\mathbb{S}^1)$ and $\|\mathbf{e}_l\|_{H^{q-r}}^2 = (1 + l^2)^{q-r}$. Therefore, if inequality (38) is satisfied, we get

$$\|P_n(u_1, \dots, u_n)u_0\|_{H^{q-r}}^2 \leq C_n^2 \sum_{l \in \mathbb{Z}} (1 + l^2)^{q-r} \left(\sum_{m_0 + \dots + m_n = l} \prod_{j=0}^n (1 + m_j^2)^{r/2} |\hat{u}_j(m_j)| \right)^2.$$

Observe now that, given smooth functions v_0, v_1, \dots, v_n , we have

$$\widehat{v_0 \cdots v_n}(l) = \sum_{m_0 + \dots + m_n = l} \hat{v}_0(m_0) \cdots \hat{v}_n(m_n).$$

In addition $H^{q-r}(\mathbb{S}^1)$ is a Banach algebra, since $q - r \geq 1$, cf. [30, Theorem 2.8.3]. Consequently there exists a constant $C'_{n,p-r}$ such that

$$\begin{aligned} \sum_{l \in \mathbb{Z}} (1 + l^2)^{q-r} \left| \sum_{m_0 + \dots + m_n = l} \hat{v}_0(m_0) \cdots \hat{v}_n(m_n) \right|^2 \\ \leq C_{n,q-r}'^2 \|v_0\|_{H^{q-r}}^2 \cdots \|v_n\|_{H^{q-r}}^2 \end{aligned}$$

for all smooth functions v_0, v_1, \dots, v_n . Putting now

$$\hat{v}_j(m_j) := (1 + m_j^2)^{r/2} |\hat{u}_j(m_j)|, \quad j = 0, \dots, n$$

in this last inequality and using the fact that the functions with Fourier coefficient $\hat{v}(m)$ and $|\hat{v}(m)|$ have the same H^{q-r} norm, we obtain

$$\|P_n(u_1, \dots, u_n)u_0\|_{H^{q-r}} \leq C_n C'_{n,q-r} \|u_0\|_{H^q} \cdots \|u_n\|_{H^q},$$

which achieves the proof. \square

Finally, we will need to define a condition on the symbol of the Fourier multiplier P in order that the operators P_n are bounded. For this purpose, the following lemma will be useful.

Lemma 32. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function of class C^{n-1} with $n \geq 1$. Suppose that $f^{(n-1)}$ is absolutely continuous and that there exists $C > 0$ and $\alpha \geq 1$ such that*

$$|f^{(n)}(\xi)| \leq C(1 + \xi^2)^{(\alpha-1)/2},$$

almost everywhere. Then

$$\left| \sum_{p=0}^n (-1)^p \sum_{\substack{I \subset \{1, \dots, n\}, \\ |I|=p}} f\left(m_0 + \sum_{j \in I} m_j\right) \right| \leq C(n+1)^{\alpha-1} (1+m_0^2)^{(\alpha-1)/2} \prod_{j=1}^n (1+m_j^2)^{\alpha/2},$$

for all $m_0, m_1, \dots, m_n \in \mathbb{R}$.

Proof. Let g_k be the sequence of functions defined inductively by

$$g_1(\xi) = f(\xi + m_1) - f(\xi), \quad g_{k+1}(\xi) = g_k(\xi + m_{k+1}) - g_k(\xi).$$

Then, we have

$$g_n(\xi) = (-1)^n \sum_{p=0}^n (-1)^p \sum_{\substack{I \subset \{1, \dots, n\}, \\ |I|=p}} f\left(\xi + \sum_{j \in I} m_j\right)$$

Let K be the convex hull of the points $m_0 + \sum_{j \in I} m_j$, for all subset I of $\{1, \dots, n\}$. Then

$$\max_{t \in K} (1 + \xi^2) \leq (1+n)^2 \prod_{j=0}^n (1 + m_j^2)$$

and therefore, $|f^{(n)}(\xi)|$ is bounded almost everywhere on K by

$$M := C(n+1)^{\alpha-1} \prod_{j=0}^n (1 + m_j^2)^{(\alpha-1)/2}.$$

Since $f^{(n-1)}$ is absolutely continuous, this leads to

$$\left| g_1^{(n-1)}(\xi) \right| \leq M |m_1|, \quad \forall \xi \in K.$$

Let I_n be the segment $[m_0, m_0 + m_n]$ and for $p = 1, \dots, n-1$, define inductively I_{n-p} as the convex hull of

$$\{s \in [\xi, \xi + m_n]; \xi \in I_{n-p+1}\}.$$

Notice that $I_n \subset I_{n-1} \subset \dots \subset I_1 \subset K$. For $p = 1, \dots, n-1$, let

$$M_{n-p} := \max_{I_{n-p+1}} \left| g_{n-p}^{(p)} \right|.$$

Notice that $M_1 \leq M |m_1|$ and that by the mean value theorem

$$M_{n-p} \leq |m_{n-p}| M_{n-(p+1)}, \quad p = 1, \dots, n-2$$

Therefore we have

$$|g_n(m_0)| \leq M \prod_{j=1}^n |m_j|,$$

which achieves the proof, since $|m_j| \leq \sqrt{1 + m_j^2}$. □

APPENDIX B. CONTINUITY LEMMAS

In this section we provide some continuity properties of the composition mapping in Sobolev spaces. Given Fréchet spaces X and Y , let $\mathcal{L}(X, Y)$ denote the space of all continuous linear operators from X into Y .

Lemma 33. *Let X, Y be Fréchet spaces and let G be a metric space. Given $F : G \times X \rightarrow Y$, assume that*

$$(39) \quad \begin{cases} F(g, \cdot) \in \mathcal{L}(X, Y) & \text{for all } g \in G, \\ F(\cdot, x) \in C(G, Y) & \text{for all } x \in X. \end{cases}$$

Then $F \in C(G \times X, Y)$.

Proof. Fix $(g_0, x_0) \in G \times X$ and pick a sequence (g_n, x_n) in $G \times X$ such that $\lim_n (g_n, x_n) = (g_0, x_0)$. Let further V denote a neighbourhood of $F(g_0, x_0)$ in Y . We set

$$B_n := F(g_n, \cdot) \in \mathcal{L}(X, Y), \quad n \in \mathbb{N}.$$

Then, given $x \in X$, we have

$$\lim_n B_n(x) = \lim_n F(g_n, x) = F(g_0, x).$$

Hence $\{B_n(x); n \in \mathbb{N}\}$ is bounded in Y . Invoking the *uniform boundedness principle* in Fréchet spaces (see [10, Theorem II.11]), we deduce that the family $\{B_n; n \in \mathbb{N}\}$ is equicontinuous. In particular there is a neighbourhood U of x_0 in X such that $B_n(U) \subset V$ for all $n \in \mathbb{N}$. But $\lim_n x_n = x_0$. Hence there is a $n_0 \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq n_0$. This implies that

$$B_n(x_n) = F(g_n, x_n) \in V \quad \text{for all } n \geq n_0.$$

Thus F is continuous in (g_0, x_0) . \square

Lemma 34. (i) *Given $q \in (3/2, \infty)$, the mapping*

$$(40) \quad F : \mathcal{D}^q(\mathbb{S}^1) \times H^1(\mathbb{S}^1) \rightarrow H^1(\mathbb{S}^1), \quad F(\varphi, v) := v \circ \varphi$$

is continuous.

(ii) *Given $q, \rho \in (3/2, \infty)$ with $q \geq \rho$, the restriction of F satisfies*

$$F \in C(\mathcal{D}^q(\mathbb{S}^1) \times H^\rho(\mathbb{S}^1), H^\rho(\mathbb{S}^1)) \cap C(\mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1), H^q(\mathbb{S}^1)).$$

Moreover there exists a continuous increasing function $C_{q,\rho} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

$$(41) \quad \|R_\varphi\|_{\mathcal{L}(H^\rho(\mathbb{S}^1))} \leq C_{q,\rho} \left(\|\varphi\|_{H^q(\mathbb{S}^1)} \right)$$

for each $\varphi \in \mathcal{D}^q(\mathbb{S}^1)$.

Proof. (a) Let $q > 3/2$ be given. By Sobolev's embedding theorem we know that $\mathcal{D}^q(\mathbb{S}^1) \hookrightarrow C^1(\mathbb{S}^1)$. Hence the chain rule ensures that F is well-defined, i.e. $F(\varphi, v) \in H^1(\mathbb{S}^1)$ for all $(\varphi, v) \in \mathcal{D}^q(\mathbb{S}^1) \times H^1(\mathbb{S}^1)$. Moreover, fixing $\varphi \in \mathcal{D}^q(\mathbb{S}^1)$, we have

$$F(\varphi, \cdot) \in \mathcal{L}(H^1(\mathbb{S}^1)).$$

(b) Let now $v \in H^1(\mathbb{S}^1)$ be fixed. We are going to show that

$$F(\cdot, v) \in C(\mathcal{D}^q(\mathbb{S}^1), H^1(\mathbb{S}^1)).$$

For this pick $\varphi_0 \in \mathcal{D}^q(\mathbb{S}^1)$ and $\varepsilon > 0$. By Sobolev's embedding theorem, the function v is uniformly continuous. Thus there is a $\delta > 0$ such that

$$|v(x) - v(y)| < \varepsilon \quad \text{for all } |x - y| < \delta.$$

Next let j denote the embedding constant of $H^q(\mathbb{S}^1) \hookrightarrow C^1(\mathbb{S}^1)$ and choose $\varphi \in \mathcal{D}^q(\mathbb{S}^1)$ such that $\|\varphi_0 - \varphi\|_{H^q} < \delta/j$. Then

$$|\varphi_0(x) - \varphi(x)| \leq j \|\varphi_0 - \varphi\|_{H^q} < \delta \quad \text{for all } x \in \mathbb{S}^1.$$

Thus we get

$$(42) \quad \|v \circ \varphi_0 - v \circ \varphi\|_{L^2}^2 = \int_{\mathbb{S}^1} |v(\varphi_0(x)) - v(\varphi(x))|^2 dx \leq \varepsilon^2.$$

To estimate $D(v \circ \varphi_0 - v \circ \varphi)$ in L^2 , we first remark that it is no restriction to assume that $\delta \leq 1$. Writing now $K := j(\|\varphi_0\|_{H^q} + 1)$ and

$$B_q(\delta) := \mathcal{D}^q(\mathbb{S}^1) \cap \mathbb{B}_{H^q}(\varphi_0, \delta),$$

we have that

$$(43) \quad \|\varphi_x\|_{L^\infty} \leq j \|\varphi\|_{H^q} \leq j(\|\varphi_0\|_{H^q} + 1) = K \quad \text{for all } \varphi \in B_q(\delta).$$

Furthermore, letting $m(\varphi) := \|1/\varphi_x\|_{L^\infty}$ for $\varphi \in \mathcal{D}^q(\mathbb{S}^1)$, we have

$$\|f \circ \varphi\|_{L^2}^2 \leq m(\varphi) \|f\|_{L^2}^2 \quad \text{for all } f \in L^2(\mathbb{S}^1).$$

Note also that by shrinking $\delta > 0$, we may assume that

$$(44) \quad m(\varphi) \leq 2m(\varphi_0) \quad \text{for all } \varphi \in B_q(\delta).$$

We now proceed as follows. First we have

$$(45) \quad \|D(v \circ \varphi_0 - v \circ \varphi)\|_{L^2}^2 \leq \|v_x \circ \varphi_0 \cdot \varphi_{0x} - v_x \circ \varphi_0 \cdot \varphi_x\|_{L^2}^2 + \|v_x \circ \varphi_0 \cdot \varphi_x - v_x \circ \varphi \cdot \varphi_x\|_{L^2}^2.$$

For the first term of the right-hand side of (45), we find

$$(46) \quad \begin{aligned} \|v_x \circ \varphi_0 \cdot \varphi_{0x} - v_x \circ \varphi_0 \cdot \varphi_x\|_{L^2}^2 &= \int_{\mathbb{S}^1} |v_x(\varphi_0(x))|^2 |\varphi_{0x}(x) - \varphi_x(x)|^2 dx \\ &\leq \|\varphi_{0x} - \varphi_x\|_{L^\infty}^2 \int_{\mathbb{S}^1} |v_x(\varphi_0(x))|^2 dx \\ &\leq j^2 \|\varphi_0 - \varphi\|_{H^q}^2 m(\varphi_0) \|v\|_{H^1}^2 = \delta^2 m(\varphi_0) \|v\|_{H^1}^2. \end{aligned}$$

To estimate the second term in (45), choose $w \in C^2(\mathbb{S}^1)$ such that

$$(47) \quad \|v - w\|_{H^1} \leq \sqrt{\frac{1}{3 m(\varphi_0)}} \frac{\varepsilon}{K}.$$

Then we have

$$\begin{aligned}
\|v_x \circ \varphi_0 - v_x \circ \varphi\|_{L^2}^2 &\leq \|v_x \circ \varphi_0 - w_x \circ \varphi_0\|_{L^2}^2 + \|w_x \circ \varphi_0 - w_x \circ \varphi\|_{L^2}^2 \\
&\quad + \|w_x \circ \varphi - v_x \circ \varphi\|_{L^2}^2 \\
&\leq (m(\varphi_0) + m(\varphi)) \|v_x - w_x\|_{L^2}^2 \\
&\quad + \int_{\mathbb{S}^1} |w_x(\varphi_0(x)) - w_x(\varphi(x))|^2 dx \\
&\leq 3m(\varphi_0) \|v_x - w_x\|_{L^2}^2 + \|w_{xx}\|_{L^\infty}^2 \int_{\mathbb{S}^1} |\varphi_0(x) - \varphi(x)|^2 dx,
\end{aligned}$$

where we also employed the mean value theorem and (44) to derive the last estimate. Invoking (43) and (47), we get

$$(48) \quad \|v_x \circ \varphi_0 \cdot \varphi_x - v_x \circ \varphi \cdot \varphi_x\|_{L^2} \leq \varepsilon + \delta K \|w_{xx}\|_{L^\infty}$$

for all $\varphi \in B_q(\delta)$. Combining (42), (45), (46), and (48), we arrive at the following estimate

$$(49) \quad \|v \circ \varphi_0 - v \circ \varphi\|_{H^1} \leq 2\varepsilon + \delta \left(\sqrt{m(\varphi_0)} \|v\|_{H^1} + K \|w_{xx}\|_{L^\infty} \right)$$

for all $\varphi \in B_q(\delta)$. Shrinking $\delta > 0$, we get from (49) that

$$\|v \circ \varphi_0 - v \circ \varphi\|_{H^1} \leq 3\varepsilon$$

for all $\varphi \in B_q(\delta)$. Thus $F(\cdot, v)$ is continuous in $\varphi_0 \in \mathcal{D}^q(\mathbb{S}^1)$. Invoking Lemma 33, we find that $F \in C(\mathcal{D}^q(\mathbb{S}^1) \times H^1(\mathbb{S}^1), H^1(\mathbb{S}^1))$. This completes the proof of part (i).

(c) To prove the second assertion, fix first $\varphi \in \mathcal{D}^q(\mathbb{S}^1)$ and observe that $F(\varphi, \cdot) = R_\varphi$. Given $k \in \mathbb{N}$ with $1 \leq k \leq q$, a direct calculation and an application of Sobolev's embedding theorem shows that there exists a continuous, increasing function $C_k : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, such that

$$\|R_\varphi\|_{\mathcal{L}(H^k(\mathbb{S}^1))} \leq C_k \left(\|\varphi\|_{H^q(\mathbb{S}^1)} \right).$$

Assume now that $\rho > 3/2$ is not an integer and belongs to $(l-1, l)$ for some $l \in \mathbb{N}$. Then, there exists $\theta \in (0, 1)$ such that $\theta(l-1) + \theta l = \rho$. Since the scale of Sobolev spaces $\{H^\sigma(\mathbb{S}^1); \sigma \in \mathbb{R}\}$ is stable under complex interpolation, cf. Theorems 7.4.4 and 7.4.5 in [31], we get by interpolation that there exists a positive constant K_ρ such that

$$\|R_\varphi\|_{\mathcal{L}(H^\rho(\mathbb{S}^1))} \leq K_\rho \|R_\varphi\|_{\mathcal{L}(H^{l-1}(\mathbb{S}^1))}^\theta \|R_\varphi\|_{\mathcal{L}(H^l(\mathbb{S}^1))}^{1-\theta}.$$

Letting now

$$C_{q,\rho}(t) = K_\rho C_{l-1}(t)^\theta C_l(t)^{1-\theta},$$

we obtain (41).

(d) By assumption we have that $\rho > 3/2$. Hence $\mathcal{D}^\rho(\mathbb{S}^1)$ is a topological group and $F(\cdot, v) \in C(\mathcal{D}^q(\mathbb{S}^1), H^\rho(\mathbb{S}^1))$ for any $v \in H^\rho(\mathbb{S}^1)$ (see [11, page 107]). By virtue of Lemma 33, we get that

$$F \in C(\mathcal{D}^q(\mathbb{S}^1) \times H^\rho(\mathbb{S}^1), H^\rho(\mathbb{S}^1)).$$

The last assertion $F \in C(\mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1), H^q(\mathbb{S}^1))$ is now clear. \square

Remarks 35. (a) Observe that there is a gap in the range of admissible Sobolev norms in Lemma 34 between assertion (i) and the second assertion (ii).

(b) Continuity properties of composition operators in low Sobolev spaces have recently been investigated in [9]. However the focus of the studies in [9] is so to speak opposite to that of (40), since more regularity of the first factor v in $v \circ \varphi$ is assumed, whereas we impose additional regularity on the second factor φ .

(c) The higher the spatial regularity in the group $\mathcal{D}^q(\mathbb{S}^1)$ and the Lie algebra $H^q(\mathbb{S}^1)$, the better the regularity of the mapping F in Lemma 34, cf. [11]. However, we are not aware of better regularity of F than (40). Finally, we remark that the continuity of F is sufficient for our purposes.

We conclude this Appendix by an auxiliary result on the boundedness of the inverse of the right translation in $H^q(\mathbb{S}^1)$.

Lemma 36. *Let $q > 3/2$ be given and assume that $B \subset \mathcal{D}^q(\mathbb{S}^1)$ is bounded in $H^q(\mathbb{S}^1)$. Then*

$$(50) \quad \sup_{\varphi \in B} \|R_{\varphi^{-1}}\|_{\mathcal{L}(H^q)} < \infty.$$

Proof. By the uniform boundedness principle it suffices to show that

$$\sup_{\varphi \in B} \|v \circ \varphi^{-1}\|_{H^q} < \infty \quad \text{for all } v \in H^q(\mathbb{S}^1).$$

If $q > 3/2$ is an integer this follows by a direct calculation and an application of Sobolev's embedding theorem. If q is not an integer, let $k \in \mathbb{N}$ and $\sigma \in (0, 1)$ such that $q = k + \sigma$. Again a direct calculation shows that it suffices to show that

$$\sup_{\varphi \in B} \|(\partial^k v) \circ \varphi^{-1}\|_{H^\sigma} < \infty \quad \text{for all } v \in H^q(\mathbb{S}^1),$$

which follows from the fact that $\partial^k v \in H^\sigma(\mathbb{S}^1)$ and by using the intrinsic norm

$$\left(\|w\|_{L^2}^2 + \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \frac{|w(x) - w(y)|^2}{|x - y|^{1+2\sigma}} dx dy \right)^{1/2}$$

for $w \in H^\sigma(\mathbb{S}^1)$, cf. Section 2.2.2 in [30]. \square

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